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Appendix: Details on Experiments (Counting and Estimating Lattice Points)

Here we present details for experiments with the Gaussian estimation of Barvinok and Hartigan. As described in the article, we needed the maximum objective values and the unique optimal solutions of the strictly concave optimization problems from Theorems 1 and 2 in order to calculate the Gaussian estimations. These values were obtained using LOQO and the rest of the calculation was implemented inside MAPLE version 12. There is no need for any sophisticated computation, one can easily calculate the covariance matrices and the lattice index using a Smith normal form calculation. Overall the evaluation step takes a negligible amount of time in all instances, so we do not record any time of computation.

A.1 General knapsacks

The first set of problems consist of knapsack-constrained simplices. The majority of the instances were randomly generated with coefficients between 1 and 20, but we also used a few famous hard knapsack problems from [1] and a few instances with coefficients in arithmetic progressions. The presence of these two kinds of tests we hope is useful on investigating how suitable are these techniques for feasibility detection and to test the sensitivity of the estimate to the variation of right-hand-side vectors. Tables 1 and 2 present the data used here and the results. On the line below any instance $ax = b$ we report the relative error of the estimation after 5000 samples and the true number of lattice points.

In Table 2 we present the results of estimation using the maximum entropy Gaussian estimate: Although there are a few difficult cases, the majority of instances has relative error less than one.

The Aardal-style knapsacks (examples 13, 14 in the tables) [1] show a dramatically poor behavior which seems to be correlated to the objective function value γ of the convex function we maximize. By Theorem 1 this number is a close indicator of success to generate a lattice point during sampling. For example, problem number 7 with right-hand-side 22382774 has no solutions, but if we increase it by one unit there are 218842 integer solutions. In both cases the values for γ and probability parameters were identical ($\gamma = 47.58769$). Through sampling we expect to require $e^{47.6} = (4.7)\times 10^{20}$ samples before finding a lattice point.

Nevertheless, for the Aardal examples, in addition to the standard convex maximization of Theorem 1 we also used the weighted version and we were surprised to see that in both instances we found an actual integral feasible solution already using the convex

Table 1. *Knapsack tests using sampling algorithm, information on relative error (best possible among three sampling tests).*

Problem	$ax = b$, sampling error, and number of solutions
#1	a=[2, 11, 18, 4, 17, 19, 6, 9, 2, 10, 16, 4, 18, 1, 15, 6, 17, 2, 8, 10, 7, 19, 7, 10, 14] b=5000 Error=0.0309433 numsols=856964145813382641466348392445250609474280
#2	a=[5, 10, 10, 2, 8, 20, 15, 2, 9, 9, 7, 4, 12, 13, 19] b=34 Error=0.01329 numsols=2056
#3	a=[10, 15, 9, 12, 11, 20, 8, 9, 17, 18, 11, 20, 13, 8, 17] b=19 Error=0 numsols=6
#4	a=[15, 13, 6, 2, 12, 4, 6, 5, 17, 8, 5, 2, 18, 20, 11] b=500 Error=0.06840749 numsols=242818430132799
#5	a=[19, 14, 18, 15, 8, 10, 14, 12, 9, 13, 16, 1, 6, 13, 14] b=500 Error=0.290757 numsols=348929541791 1
#6	a=[7, 7, 3, 14, 15, 15, 19, 12, 19, 8, 3, 17, 17, 3, 5] b= 50 Error=0.015823 numsols=8438
#7	a=[20601, 40429, 40429, 45415, 53725, 61919, 64470, 69340, 78539, 95043] b=22382775 Error=1 numsols=218842
#8	a=[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] b=255 Error=0.131238 numsols=71660385050
#9	a=[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] b=24 Error=0.019237 numsols=1380
#10	a=[9, 1, 19, 4, 6, 3, 4, 10, 8, 2] b=50 Error=0.0159488 numsols=47761
#11	a=[16, 2, 11, 15, 11, 3, 5, 14, 7, 2] b=500 Error=0.037474 numsols=65552595947
#12	a=[8, 10, 9, 17, 2, 9, 3, 2, 5, 20] b=25 Error=0.00589 numsols=267
#13	a=[12137, 24269, 36405, 36407, 48545, 60683] b=58925135 Error=1 numsols=2
#14	a=[12223, 12224, 36674, 61119, 85569] b= 89643482 Error=1 numsols=1
#15	a=[1, 2, 3, 4, 5, 6] b=25 Error=0.03970 numsols=612
#16	a=[9, 10, 17, 5, 2] b=73 Error=0.00208 numsols=209
#17	a=[9, 11, 14, 5, 12] b=26 Error=0.015148 numsols=3
#18	a=[5, 3, 1, 4, 2] b=15 Error=0.02186 numsols=84
#19	a=[5, 13, 2, 8, 3] b=17 Error=0.02991 numsols=12
#20	a=[8, 12, 11] b=50 Error=0.1535 numsols=1

optimization procedure. The weights were generated at random, so this raises the issue of understanding whether for good weights one can find integer solutions systematically.

Table 2. *Knapsack tests for Gaussian estimation, we report the relative error.*

Problem	$ax = b$, Gaussian estimate relative error, and number of solutions
#1	a=[2, 11, 18, 4, 17, 19, 6, 9, 2, 10, 16, 4, 18, 1, 15, 6, 17, 2, 8, 10, 7, 19, 7, 10, 14] b=5000 Error = 0.00334 numsols =8569641458133826414663483924452506094742800
#2	a=[5, 10, 10, 2, 8, 20, 15, 2, 9, 9, 7, 4, 12, 13, 19] b=34 Error = 0.00585 numsols =2056
#3	a=[10, 15, 9, 12, 11, 20, 8, 9, 17, 18, 11, 20, 13, 8, 17] b=19 Error = 0.97855 numsols =6
#4	a=[15, 13, 6, 2, 12, 4, 6, 5, 17, 8, 5, 2, 18, 20, 11] b=500 Error = 0.79371 numsols =242818430132799
#5	a=[19, 14, 18, 15, 8, 10, 14, 12, 9, 13, 16, 1, 6, 13, 14] b=500 Error = 0.00618 numsols =3489295417911
#6	a=[7, 7, 3, 14, 15, 15, 19, 12, 19, 8, 3, 17, 17, 3, 5] b= 50 Error = 0.96326 numsols =8438
#7	a=[20601, 40429, 40429, 45415, 53725, 61919, 64470, 69340, 78539,95043] b=22382775 Error = 118163207.8 numsols =218842
#8	a=[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] b=255 Error = 0.89587 numsols =71660385050
#9	a=[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] b=24 Error = 0.02503 numsols =1380
#10	a=[9, 1, 19, 4, 6, 3, 4, 10, 8, 2] b=50 Error = 0.01428 numsols =47761
#11	a=[16, 2, 11, 15, 11, 3, 5, 14, 7, 2] b=500 Error = 3.16010 numsols =65552595947
#12	a=[8, 10, 9, 17, 2, 9, 3, 2, 5, 20] b=25 Error = 0.04816 numsols =267
#13	a=[12137, 24269, 36405, 36407, 48545, 60683] b=58925135 Error = 2634013736 numsols =2
#14	a=[12223, 12224, 36674, 61119, 85569] b= 89643482 Error = 95905325.28 numsols =1
#15	a=[1, 2, 3, 4, 5, 6] b=25 Error =0.02599 numsols =612
#16	a=[9, 10, 17, 5, 2] b=73 Error = 0.03213 numsols =209
#17	a=[9, 11, 14, 5, 12] b=26 Error = 0.81445 numsols =3
#18	a=[5, 3, 1, 4, 2] b=15 Error =0.02923 numsols =84
#19	a=[5, 13, 2, 8, 3] b=17 Error = 0.40546 numsols =12
#20	a=[8, 12, 11] b=50 Error = 1.0890 numsols =1

A.2 Contingency tables and network flows

We continue our investigations with the class of *multiway contingency tables*. A *d-table of size* $(n_1, \ldots, n_d)$ is an array of non-negative integers $v = (v_{i_1,\ldots,i_d})$, $1 \le i_j \le n_j$. For $0 \le m < d$, an *m-marginal* of v is any of the $\binom{d}{m}$ possible m-tables obtained by summing the entries over all but m indices.

Contingency tables appear naturally in statistics and operations research under various names such as *multi-way contingency tables*, or *tabular data*. Table counting has several applications in statistical analysis, in particular for independence testing, and has been the focus of much research (see [38] and the extensive list of references therein). Given a specified collection of marginals for d-tables of size $(n_1, \ldots, n_d)$ (possibly together with specified lower and upper bounds on some of the table entries) the associated *multi-index transportation polytope* is the set of all non-negative *real valued* arrays satisfying these marginals and entry bounds. The counting problem can be formulated as that of counting the number of integer points in the associated multi-index transportation polytope.

First we treat *two-way contingency tables*. The polytope defined by a two-way contingency table is called the *transportation polytope*. We present the results in Table 3. This family of polytopes has been studied in their lattice points by several authors [8, 14, 16, 32, 72] and thus are good for testing accuracy of the estimation. We used several examples of 4×4 transportation polytopes, as presented in the table below. In all cases the relative error was very similar to the performance with simplices, but in this case we are simply using the Gaussian estimator proposed by Barvinok and Hartigan [15]. The results are good. It should be remarked that sampling did not do well at all for these instances. In fact, in all problems with multiple constraints (non-knapsack) we had a bad performance of sampling, thus we aggregated the constraints in a few instances, but that did not improve the behavior.

Another closely related set of instances are those coming from flows on networks, which are the lattice points of *flow polytopes*. Again, testing accuracy of the estimation we looked at a simple network on the complete 4-vertex and the complete 5-vertex tournaments (directed complete graphs). Consider the directed complete graph K_n. We assign a number to each node of the graph. Then, we orient the arcs from the node of smaller index to the node of bigger index. Let N be the node set of the complete graph K_n, let w_i be a weight assigned to node i for $i = 1, 2, \ldots, n$, and let A be the arc set of K_l. Then, we have the following constraints, with as many variables as arcs:

$$\sum_{(j,i)\text{arc enters} i} x_{ji} - \sum_{(i,j)\text{arc has tail} i} x_{ij} = w_i \quad \forall i \in N,$$

$$x_{ij} \ge 0 \ \forall (i,j) \in A.$$

These equalities and inequalities define a polytope and this polytope is a special case of network flow polytope. Some tests for the complete graphs K_4 and K_5, with different weight vectors, are shown in Table 4.

Next we tested more complicated situations. We consider 3-way transportation polytopes where the constraints are given by 1-margins. In the first half of Table 5 one can see the behavior is like that of the 2-way transportation polytopes in Table 3. The sec-

Table 3. *Testing for 4×4 transportation polytopes.*

Margins	exact # of lattice points	Error on estimation
[1,1,1,1], [1,1,1,1]	24	0.36277
[300,300,300,300], [300,300,300,300]	2026969959692633775l	0.12179
[220, 215, 93, 64], [108, 286, 71, 127]	1225914276768514	0.06062
[109, 127, 69, 109], [119, 86, 108, 101]	993810896945891	0.11770
[72, 67, 47, 96], [70, 70, 51, 91]	25387360604030	0.08014
[179909, 258827, 224919, 61909], [190019, 90636, 276208, 168701]	1357102606340183816466829663506589992315207 9	0.03177
[229623, 259723, 132135, 310952], [279858, 170568, 297181, 184826]	6469113954592966452000040008040032433711548 62	0.07002
[249961, 232006, 150459, 200438], [222515, 130701, 278288, 201360]	3197202496901114378872292554878478453104634 75	0.08213
[140648, 296472, 130724, 309173], [240223, 223149, 218763, 194882]	3227735608210088564172702759505991070612636 25	0.04441
[65205, 189726, 233525, 170004], [137007, 87762, 274082, 159609]	697752372074002424105607512161102113957691 9	0.01276
[251746, 282451, 184389, 194442], [146933, 239421, 267665, 259009]	8613163432806490495932361321550391906820276 14	0.08974
[138498, 166344, 187928, 186942], [228834, 138788, 189477, 122613]	633131914143428277545665313645333785889864 67	0.08647
[20812723, 17301709, 21133745, 27679151], [28343568, 18410455,19751834, 20421471]	6657115555677923898789089936246293791879698801797211690688279 51	0.99997
[15663004, 19519372, 14722354, 22325971], [17617837, 25267522, 20146447, 9198895]	63292704423941655080293971395348848807454253204720526472462015	0.99987
[13070380, 18156451, 13365203, 20567424], [12268303, 20733257, 17743591, 14414307]	43075357146173570492117291685601604830544643769252831337342557	0.99925

Table 4. *Gaussian estimation for four and five network problems.*

Weights on Nodes	Number of Lattice points	Error
[−, 2569, −3820, 1108, 5281]	14100406254	0.1179212818
[−3842, −3945, 6744, 1043]	1906669380	0.07844816179
[−47896, −30744, 46242, 32398]	19470466783680	0.08633129132
[−54915, −97874, 64165, 88624]	106036300535520	0.1903691374
[−69295, −62008, 28678, 102625]	179777378508547	0.2495250224
[−3125352, −6257694, 926385, 8456661]	344414801726951O1274	0.9847314633
[−12, −8, 9, 7, 4];	14805;	0.9331835846
[−125, −50, 75, 33, 67];	6950747024	0.8060985686
[−763, −41, 227, 89, 488]	222850218035543	0.4040804721
[−52541, −88985, 1112, 55665, 84749]	39971216842426033014442665332	0.9727043353
[−45617, −46855, 24133, 54922, 13417]	39971216842426033014442665332	0.9999790656
[−69295, −62008, 28678, 88725, 13900]	65348330279808617817420057	0.9998969873
[−8959393, −2901013, 85873, 533630, 11240903]	6817997013081449330251623043931489475270	0.9173160458

Table 5. *Testing for 3-way transportation polytopes. The top half deals with 1-margins and the second half with 2-margins.*

Margins	# of lattice points	Error
[38,26,87], [69,11,71], [77,74]	2626216761994	0.02956
[381,264,871], [690,123,703], [672,844]	969328784998192447450409	0.02201
[179909, 258827, 224919], [190019, 276208, 197428], [331827,331828]	10996570128188103700571192439719329377965299537177538734365	0.10601
[31,22,87], [50,13,77], [42,87,11]	8846838772161591	0.00185
[531,992,787], [750,913,577], [742,587,911]	75262943725025193225827940796161419321644384	0.11337
[11531,9922,13787], [13750,9913,11577], [9742,13587,11911]	131537460708108801553237287957587623890623217109985417250150842074021	0.12606
[1153100, 992200, 1378700], [1375000, 991300, 1157700], [974200, 1358700, 1191100]	13047034222952410155948716772275006229691435459379246218796390438655039 21583490166934483708040820516427400011	0.12628
$\begin{bmatrix} 10002 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$, $\begin{bmatrix} 1000003 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$, $\begin{bmatrix} 1000003 & 3 \\ 3 & 3 \\ 3 & 3 \end{bmatrix}$	33	0.10437
$\begin{bmatrix} 282 & 199 & 212 \\ 21 & 14 & 9 \\ 1 & 2 & 2 \end{bmatrix}$, $\begin{bmatrix} 107 & 80 & 169 \\ 197 & 135 & 54 \end{bmatrix}$, $\begin{bmatrix} 329 & 364 \\ 23 & 21 \\ 4 & 1 \end{bmatrix}$	441	1.10058
(3 copies of) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	12	12.3843

ond half has three 3-way transportation polytopes given by sets of 2-margins. Due to the difference of error the examples indicate that the method of estimation seems to have its best performance when the number of variables is much larger than the number of constraints. This is particularly evident in the last two examples of Table 5. The first considers a 2-margin $2 \times 3 \times 3$ transportation polytope taken from [45]. Second, the $3 \times 3 \times 3$ analogue of the Birkhoff polytope has clear inaccuracy. Again, as we observed before, when the computation involves large entries the convex optimization problem becomes much harder to solve.

In [29] the authors introduced a class of small but rather difficult problems for the purpose of detecting feasibility. Roughly speaking these are problems of the form $Ax = b, x \in \{0, 1\}$ where A is an $m \times 10(m - 1)$ matrix and the entries of A are integers between 0 and 99. The i-th entry of b is the sum of the corresponding row of A, divided by 2 and rounded to the next integer value. We investigated the performance of estimation in this kind of problems. We generated all instances using the generator available at http://did.mat.uni-bayreuth.de/~alfred/marketsplit.html. We only investigated feasible instances with $m = 2$, $m = 3$ and $m = 4$. A very high percentage of instances produced are in fact infeasible.

Unfortunately, when we tried the sampling technique in the six feasible instances we found that most of the the time sampling never found a single feasible solution. The second test was made using the Gaussian estimation and it is presented in Table 6. We list the predicted number of solutions and the true number of solutions, if any. The top part of the table consists of infeasible problems and there the prediction is consistent with that fact.

Table 6. *Testing gaussian estimation on feasible market-split problems. The problems on the top are infeasible and those in the block below have only one integer solution. Thus, with one obvious exception, the estimated number of solutions is of comparable magnitude.*

Constraints (of the form $L_i x = m_i$)	predicted v.s. real number of solutions
$L1 := [73,93,88,63,65,27,25,6,31,70]; m1 := 270;$ $L2 := [17,27,32,35,36,63,64,37,91,93] : m2 := 247$	0.0271 / 0
$L1 := [30,29,94,77,46,10,63,40,38,37]; m1 := 232;$ $L2 := [16,92,10,40,29,16,51,11,31,8]; m2 := 152;$	0.0435 / 0
$L1 := [10,2,88,39,33,42,48,67,25,74,86,34,76,41,20,32,42,78,26,63]; m1 := 463;$ $L2 := [15,94,80,1,21,99,54,48,10,46,71,20,0,60,11,33,2,59,52,79]; m2 := 427;$ $L3 := [85,39,14,61,80,86,46,23,16,24,86,31,19,18,85,40,69,39,40,79]; m3 := 490;$	0.0954 / 0
$L1 := [66,43,98,18,69,4,40,3,72,41,20,52,42,34,60,70,27,43,64,88]; m1 := 477;$ $L2 := [98,17,44,94,68,84,77,70,43,77,3,61,72,53,79,41,9,71,97,81]; m2 := 619;$ $L3 := [12,69,33,6,55,45,29,82,88,93,70,39,62,67,85,31,51,14,1,46]; m3 := 489;$	0.0784 / 0
$L_1 := [29,63,85,80,64,87,22,31,5,23,96,8,14,93,23,74,78,72,0,30]; m_1 := 488;$ $L_2 := [15,44,21,83,13,49,40,13,33,46,42,62,10,80,94,26,19,68,10,24]; m_2 := 396;$ $L_3 := [43,6,84,58,51,7,84,29,79,36,11,47,33,32,30,46,33,23,11,67]; m_3 := 405;$	0.1049 / 2221
$L_1 := [39,65,40,15,2,35,74,87,33,46,51,21,86,80,70,60,31,30,1,8]; m_1 := 437;$ $L_2 := [12,23,87,46,49,16,14,88,91,39,8,31,4,0,46,58,36,73,45,21]; m_2 := 393;$ $L_3 := [19,49,94,58,29,17,70,61,47,71,21,59,46,8,58,95,76,72,36,68]; m_3 := 527;$	0.1363 / 748
$L_1 := [69,0,89,59,32,73,56,26,49,42]; m_1 := 247;$ $L_2 := [42,69,59,0,57,31,84,95,61,77]; m_2 := 287;$	0.02752 / 10
$L_1 := [46,12,3,91,76,7,39,27,95,55]; m_1 := 225;$ $L_2 := [74,85,49,39,80,22,16,25,84,11]; m_2 := 242;$	0.0334 / 8

Table 7. *Gaussian estimation for binary knapsack problems*

Constraints (of the form $Lx = m$)	# $0-1$ solutions	Error
$L := [116,192,120,401,129]; m := 236;$	1	0.75480
$L := [11,17,33,13,3,5,6,7,4]; m := 33$	9	0.11174
$L := [16,92,10,40,29,16,51,11,31,8]; m := 152;$	4	0.65327
$L := [46,12,3,91,76,7,39,27,95,55]; m = 225;$	7	0.33358
$L := [9,17,16,8,6,15,14,7,8,11]; m := 30;$	11	0.13587
$L := [1,2,3,4,5,6,7,8,9,10]; m := 12;$	13	0.07061
$L := [11,4,13,7,3,5,10,5,6,11]; m := 33;$	29	0.84788
$L := [1,3,5,7,9,11,13,15,17,19]; m := 19;$	6	0.04771
$L := [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15]; m := 15$	27	0.02667
$L := [1,3,5,7,9,11,13,15,17,19,21,23,25,27,29,31,33,35,37,39]; m = 39$	41	0.05350
$L := [1,3,5,7,9,11,\ldots,2i-1,\ldots,47,49]; m := 49$	93	0.03782
$L := [1,2,3,4,\ldots,i,\ldots,20]; m := 20;$	64	0.04844
$L := [1,2,3,4,\ldots,30]; m := 30;$	296	0.04371
$L := [1,2,3,4,\ldots,40]; m := 40;$	1113	0.04075
$L := [1,2,3,\ldots,50]; m := 50;$	3658	0.03619

Table 8. *Gaussian estimation for counting graphs with given degree sequences (feasible cases only)*

Graph description or sequence	number of graphs	Error
3-regular graphs with 6 nodes	70	0.00262
3-regular graphs with 8 nodes	19355	0.06182
3-regular graphs with 10 nodes	11180820	0.05657
3-regular graphs with 12 nodes	11555272575	0.02772
3-regular graphs with 14 nodes	19506631814670	0.01170
4-regular graphs with 6 nodes	15	0.17947
4-regular graphs with 8 nodes	19355	0.06182
4-regular graphs with 10 nodes	66462606	0.38190
4-regular graphs with 12 nodes	480413921130	0.21204
4-regular graphs with 13 nodes	52113376310985	0.07909
4-regular graphs with 14 nodes	6551246596501035	0.06839
4-regular graphs with 17 nodes	28797220460586826422720	0.02703
[5,6,1,1,1,1,1,1,1,1,1,1,1]	7392	1.15481

Finally we considered ten instances of binary knapsack problems in Table 7. The estimation we present was done using the Gaussian heuristic. This time the error is again smaller.

There is a wide variety of combinatorial configurations that can be described as the binary solutions of a problem of the form $Ax = b$. For example, for matchings of bipartite graphs the matrix A is the incidence matrix of the graph. Of course, this opens the possibility of using the Barvinok-Hartigan estimators to approximate various combinatorial enumeration problems. In this section we look carefully at the performance of the estimation in one well-studied combinatorial problem.

For experimentation we have selected the problem of estimating the number of labeled graphs with a prescribed degree sequence. Random graphs with given vertex degrees have interest as models for complex networks (see [23]). The main reasons for choosing this problem include: (1) There are well-known tests that allow us to decide when there are no solutions (see [69]). (2) Researchers have already documented this problem and there is a wide variety of formulas available for verification, e.g., regular graphs appear listed in Sloane's online encyclopedia of integer sequences (see [23] and references therein). Most of the results are collected in Table 8, but we also try comparing the estimation with non-feasible instances, i.e., non-realizable degree sequences. We found here that the estimation was truly off. For example, there are no 3-regular graphs with 13 nodes, but the Gaussian estimation predicted about 0.4445×10^{12} such graphs.

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