PROGRAM

OF THE

INTERNATIONAL SYMPOSIUM

ON

MATHEMATICAL PROGRAMMING

PRINCETON UNIVERSITY

August 14-18

1967

~~Send Zacha WP123~~ ~~[illegible]~~ WP120

~~Send Kirby WP123~~

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Cond, J. , ~~Mathematica WP 120, done!~~

MATHEMATICAL PROGRAMMING SYMPOSIUM

~~Submitted on WSP list (for Mathematica)~~

ABSTRACTS
of
CONTRIBUTED PAPERS

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A METHOD OF DECOMPOSITION

J. Abadie and M. Sakarovitch

Consider a Linear program with the following structure:

$$
(P)\quad \left[\begin{array}{l}
A^1 x^1 \qquad\qquad = b^1 \\
\quad \ddots \qquad\qquad\qquad\qquad x^1,\ldots,x^q \geqq 0 \\
\qquad A^q x^q = b^q \\
D^1 x^1 + \ldots + D^q x^q = d \\
c^1 x^1 + \ldots + c^q x^q = z \text{ (min)}
\end{array}\right.
$$

where the A^i are $m^i \times n^i$ matrices,

where the D^i are $m \times n^i$ matrices,

and x^i, b^i, c^i, b are respectively n^i, m^i, n^i, m vectors.

The idea of the method consists in partitioning the vector d into $d^1 + d^2 + \ldots d^q = d$ in such a way that if $\overline{x^i}$ is an optimal solution of:

$$
(P^i)\quad \left[\begin{array}{l}
A^i x^i = b^i \\
D^i x^i = d^i \\
c^i x^i = z^i \text{ (min)},
\end{array}\right.
$$

then $\overline{x} = (\overline{x}^1, \ldots, \overline{x^q})$ be an optimal solution of (P).

This is a very natural way of handling the problem (think of the allocation of scarce resources in a decentralized economy).

For a given partition of the vector d, one solves the P^i's, and from the value of the optimal prices one decides:

--either to stop if an optimal solution to (P) has been reached;

--or to alter the value of the d^i's in such a way that subsequent solutions of the P^i's will show a decrease in the objective function.

Two variants of the algorithm are proposed according to the way the alteration of the d^i's is computed:

1) A special set of optimal prices to the P^i's is found by solving a constrained minimization quadratic problem which is equivalent to a linear system. The changes of d^i are simple functions of these prices.
2) An auxiliary linear program using any set of optimal prices to the P^i's is solved, and its solution gives the new d^i's.

These algorithms are shown to be finite. Their efficiency is discussed.

MONDAY AFTERNOON

STOCHASTIC GEOMETRIC PROGRAMMING

M. Avriel and D. J. Wilde

In this paper we formulate and analyze geometric programming problems in which some of the parameters of the objective function and the constraints are random variables.

Optimal engineering design often involves finding values for design and operating variables which minimize combined capital and operating costs. Solving such problems in a systematic way and obtaining quantitative economic information on the interrelation between the optimal variables and given process conditions have been the stimulus for the development of geometric programming.

In this paper we introduce models for the case of engineering design by geometric programming where some of the parameters, representing unit costs or process conditions, are random variables. The designer is confronted then with selecting first fixed values for some of the variables (design variables), then observe the random parameters and finally choose values for the operating variables, such that the expected total cost is minimized. This model is analogous to the well-known two-stage linear program under uncertainty. It is shown that this geometric programming model can be formulated as a convex programming problem and in the case of random variables with a discrete probability distribution it reduces to an ordinary geometric program.

Similar to previous works in linear and nonlinear programming under uncertainty, we define several additional stochastic geometric programming problems (e.g., the wait-and-see problem) and establish a set of inequalities which provide upper and lower bounds on the solution of a two-stage geometric program.

Conventional design methods usually take into account uncertainties by overdesign, i.e., specifying fixed design and operating variables which satisfy the constraints for every possible outcome of the random parameters. It is shown that solving a certain geometric program one can obtain a fixed set of optimal design and operating variables which is permanently feasible.

TUESDAY EVENING

ON MAXIMUM MATCHING, MINIMUM COVERING, AND DUALITY

M. L. Balinski

This paper establishes a dual relationship between a maximum matching by weights on edges and a minimum weighted covering by nodes on a graph G. As such, it establishes a type of duality between a pair of linear integer programming problems. An interesting feature of this duality is that there exist optimal integer solutions which satisfy "one half" of the complementary orthogonal conditions which are necessary and sufficient for solutions to be optimal in the corresponding linear programs. The "one sidedness" of this statement tends to confirm the covering problem as more fundamental than the matching problem.

Necessary and sufficient conditions for a matching to be maximum and a covering to be minimum are given, respectively, in terms of paths and connected bipartite subgraphs of G.

WEDNESDAY AFTERNOON

MAXIMIZING STATIONARY UTILITY IN A CONSTANT TECHNOLOGY

R. Beals and T. C. Koopmans

This paper proves existence and studies properties of an infinite sequence $(x_1, x_2, \ldots)$ of scalar consumptions x_t which maximizes a quasi-concave utility function U satisfying a recursive relation

$$U(x_1, x_2, x_3, \ldots) = V(x_1, U(x_2, x_3, \ldots)),$$

subject to the constraints

$$x_t = z_{t-1} + f(z_{t-1}) - z_t \geq 0, \quad t = 1, 2, \ldots,$$

where z_t $(\geqq 0)$ is a capital stock, z_0 (> 0) is given, and $f(z_t)$ is the output produced in period t, with f strictly concave and increasing, $f(0) = 0$. A complete description of the asymptotic behavior of optimal capital and consumption sequences is obtained, including identification of the stable and unstable constant optimal sequences.

THURSDAY AFTERNOON

ON NEWTON'S METHOD IN NONLINEAR PROGRAMMING

A. Ben-Israel

Newton's method for solving nonlinear equations, was extended in [1] and [2] to rectangular systems and singular Jacobians by using generalized inverses. Extensions to operator equations and applications to least squares problems were given in [3]. This version of Newton's method was applied in [4] to the solution of nonlinear least squares problems over convex sets, in particular to nonlinear inequalities, and is applied here to problems of mathematical programming in particular to constrained least squares problems.

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[1] A. Ben-Israel: "A Modified Newton-Raphson Method for the Solution of Systems of Equations," Israel J. Math. 3 (1965), 94-98.
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[3] A. Ben-Israel: "On the Newton-Raphson Method," September 1966, Systems Research Memorandum No. 162, Northwestern University, Evanston, Ill.
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MONDAY AFTERNOON

OPTIMUM ROUTING IN CONNECTING NETWORKS OVER FINITE TIME INTERVALS

V. E. Beneš

A telephone connecting network is given, and with full information at all times about its state, routing policies are sought which minimize the number of attempted calls denied service in some finite interval. In this paper the search is pursued as a mathematical problem in the context of a standard traffic model in terms of optimal control theory and dynamic programming. Certain combinatorial properties of the network, earlier found to be the key to minimizing the loss, also turn out to be relevant here: they lead to policies which differ from optimal policies only in accepting all unblocked call attempts, and provide a "practical" solution of the problem posed. In many cases the policies found vindicate heuristic policies earlier conjectured to be optimal.

WEDNESDAY AFTERNOON

RENEWAL PROCESSES AND SOME STOCHASTIC PROGRAMMING PROBLEMS IN ECONOMICS

B. Bereanu

The following stochastic programming problem is considered

$$\underset{v}{\text{Min}}\; E(av + \underset{y}{\min}\; dy) \tag{1}$$

subject to

$$\sum_{i=1}^{y} c(\xi^i)x^i \geqq T \quad , \tag{2}$$

$$\left.\begin{array}{l} A(\xi^i)x^i \leqq b(\xi^i)+v \quad , \\ \qquad x^i \geqq 0 \quad , \end{array}\right. \tag{3}$$

$$y, \text{ positive integer } ,$$

$$v \in V. \tag{4}$$

In (1), (2), (3), a is an $1 \times p$ matrix, d is a scalar and $c(\xi^i)$, $A(\xi^i)$, $b(\xi^i)$ are respectively matrix functions of dimensions $1 \times n$, $m \times n$, and $m \times 1$, linear in the components of the variate ξ^i, and x^i is $n \times 1$; V is a given convex set in the positive ortant of R^p such that for $v \in V$, the set of solutions of (3) is not empty for any possible realization of ξ^i. T is a given number or a random variable with given distribution. It is further assumed that ξ^i are independent, identically distributed random vectors, having known probability density function, independent of v. E designates expectation.

This problem appeared in relation with some economic processes which may be viewed as the repetition of an 'operation' O until the cumulated 'return' of this operation reaches a target T. The vector of the levels of activities of the i-th performance of O, x^i, must satisfy restrictions (3) where the components of ξ^i are characteristics of certain resources and v represent additional production capacities which must be acquired via investments, <u>prior</u> to the beginning of the economic process. When v must be decided upon, ξ^i are known only through their distribution, but their realizations will be known, when x^i must be chosen, each time O takes place. The return of the 'operation' is $c(\xi^i)x^i$ and (1) is the cost of the process to be minimized. Such is the situation in certain seasonal industries where a rational plan of investments depends on the cost of the investments and the resulting reduction of the length of the campaign.

The programming problem (1), (2), (3), (4) is investigated under various assumptions concerning $A(\xi^i)$, $b(\xi^i)$, $c(\xi^i)$, V and the probability density function of ξ^i. An essential role in the treatment of this problem is played by a certain renewal process $\{\phi^i_v\}$ $(i=1,2,\ldots, v\varepsilon V)$ and results of [1], [2], [3], [4].

References

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2. __________, "On stochastic linear programming. Distribution problems: stochastic technology matrix"(to appear in Z. Wahrscheinlichkeitstheories und Verw. Gebiete).
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TUESDAY AFTERNOON

SECOND ORDER CHARACTERIZATION OF GENERALIZED POLYNOMIAL PROGRAMS

G. E. Blau and D. J. Wilde

The recent extension of Geometric Programming by Passy and Wilde to Generalized Polynomial Programming permits the solution of problems with both positive and negative coefficients, e.g. profit maximization or cost minimization. However, in this more general formulation, several important features of the primal-dual relationships of Geometric Programming are lost. Principal among these is the inability to identify the nature of the associated dual problem and hence use this dual for quick estimates and successive approximation algorithms.

It is the purpose of this paper to investigate the nature of the dual program corresponding to a well formulated generalized polynomial primal program. These dual programs are characterized by positivity for all dual variables and a stationary point at the optimum. By examining the second order behavior of the dual function in the feasible neighborhood of the stationary point, an equivalent constrained saddlepoint program is developed for dual programs satisfying certain qualifications on both the constraints and variables. Finally, a modified Arrow-Hurwicz algorithm is suggested for finding these constrained saddlepoints.

WEDNESDAY AFTERNOON

A "BRANCH AND BOUND" TYPE ALGORITHM FOR THE FIXED CHARGE LINEAR PROGRAMMING PROBLEM

P. Bod

Several practical problems in the field of the operation research lead to fixed-charge linear programming models. The standard form of such models is the following: Problem: find non-negative vectors $\underline{x} \geqq \underline{o}$ satisfying $A\underline{x} = \underline{b}$ such that

$$f(\underline{x}) = \underline{c}^*\underline{x} + \sum_i \delta_i K_i \to \min! \quad (K_i \geqq o)$$

where

$$\delta_i = \begin{cases} 1 \text{ if } x_i > o \\ o \text{ if } x_i = o \end{cases}$$

A separable concave function must be minimized on a polyhedral convex set, given by linear constraints. It follows from the basic structure of the problem that the set of the optimum solutions (if not empty) contains certainly one extreme point of the set: $L = \{\underline{x} | A\underline{x} = \underline{b};\ \underline{x} \geqq \underline{o}\}$. In addition all extreme points yield--no degeneracy assumed--a local optimum solution. To find one global optimum solution of the problem, it is sufficient to examine the set of the extreme points of the set L; denoted by C. Therefore the procedures which generate all elements of the set C yield the global optimum for the fixed-charge linear programming problem; however, such procedures require generally exceedingly large computation effort.

We are proposing an algorithm for solving the fixed-charge problem. The algorithm adapts the S.E.P. (Séparation et Évaluation Progressives) procedure, which is a "Branch and Bound" type family of procedures; given by B. Roy, P. Bertier and P. T. Nghiem.

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(1) W. M. Hirsch, G. B. Dantzig, "The Fixed Charge Problem," The RAND Corp., Paper P-648, 1954.

(2) P. Bertier, B. Roy, "Une procédure de résolution pour une classe de problemes pouvant avoir un caractere combinatoire," Bulletin du Centre International de Calcul, Rome, 1965.

TUESDAY EVENING

ORTHOGONALITY IN MATROIDS AND MATHEMATICAL PROGRAMMING

C. P. Bruter

In this paper, linear programming will be placed in its true context, the theory of matroids.

Some equivalent axiomatizations of matroids are given as well as examples. Linear programming is recalled.

In order to emphasize the fact that, for matroids and in particular for linear programming, orthogonality is a better term than duality, some results on an algebraic representation of matroids are given.

Next, the notion of an orthogonal of a matroid is introduced. That is, in some cases, algebraically justified. Remark that any matroid (any linear program) has an orthogonal. The theorem of orthogonality follows directly from the definitions. Finally, a generalization of the notion of convexity will appear.

A THEORETICAL ANALYSIS OF INPUTS TAXATION UNDER LINEAR PROGRAMMING ASSUMPTIONS

G. Casale

In our paper we state some economic effects of inputs taxation with reference to a firm operating in typical linear programming situation.

Our paper can be divided into two parts. In the first one, we state the tax influence on the volume of the output obtainable from the original resources endowment. In the second one, we determine the tax influence on the imputed value of each limited factor in order to infer some conclusions about the possible tax influence on the future policy of the firm.

At first, we introduce a standard one-product-firm linear programming model, and rigorously state the kind of tax we shall refer to. Then, briefly we recall the general optimum conditions of our linear programming model both in pre- and post-tax situation.

Afterwards we devote our attention to the tax influence on the firm's present output. At first, we consider a 2-processes 3-resources case, using a rather new diagrammatic tool. Later on, we enlarge our analysis to the general n-processes m-resources case, using simplex criterion logic. In this analysis we state rigorously the changes in output due to the tax; in particular we state that it may decrease as well as increase. Moreover we are able to show that the generally accepted theoretical statement that a firm operating in pure competition market can never enhance its output on account of any tax, may not be generally true if internal conditions of the firm give rise to a linear programming situation.

In the second part the assumption is taken that the future policy of the firm may be influenced by the present values imputed by the firm to its scarce resources. For this reason we must preliminarily study the tax influence on those values. In this connection, we make use of the dual properties of linear programming problems. In particular, considering the structure of the inequalities as they appear in dual form of our previously stated direct problem, we can compare the pre-taxation solutions frontier with the post-taxation one, and determine--in rather new diagrammatic way--the incluence of taxation on scarce resources imputed values.

At this stage of our analysis we demonstrate, among other things, that the generally accepted theoretical statement that a tax on a single factor affects in any case the marginal productivity of all factors, may not be generally ture when the firm operate in a linear programming milieu.

Finally, we state how the tax alters the firm's convenience to remove the causes which hinder the future expantion of each limited factor endowment.

We conclude our paper with few observations about the theoretical validity of our analysis.

THURSDAY AFTERNOON

ON CLASSES OF CONVEX AND PREEMPTIVE NUCLEI FOR N-PERSON GAMES

A. Charnes and K. Kortanek

In earlier papers we developed new connections between the duality theory of linear programming and solution concepts for n-person games such as the core, nucleolus, kernel of a game, and set forth a whole new class of convex and non-Archimedean solution concepts termed nuclei. These nuclei are in general independent of any topological considerations and possess uniqueness properties as well as core membership. In this paper we present more general programming formulations which involve possibly arbitrary subsets of a fixed group of permutations of subsets of coalitions in order to determine a system of linear constraints on the finite number of excess variables.

By introducing non-Archimedean elements to the base field (itself required to be ordered) we obtain preemptive orderings for coalition strength where the strength of a coalition may depend on particular types of collections of players. In a previous paper we considered only single permutations or conditions on the collection of excesses with respect to payoff vectors. We here generalize this (over the extended field) to include subsets of permutations and possibly subcollections of coalitions of players. Finally we develop (over the real field) convex solution nuclei along the lines of semi-infinite duality theory for an arbitrary number of conditions on the finite set of excess variables with respect to payoff vectors. In this manner we obtain specific examples of strictly convex nuclei which are not differentiable.

WEDNESDAY AFTERNOON

ON THE SOLUTION OF STRUCTURED LP PROBLEMS USING GENERALIZED INVERSE METHODS

R. E. Cline and L. D. Pyle

A number of methods have been formulated for solving LP problems using the Moore-Penrose generalized inverse of the constraint matrix. Although applicable to general LP problems it seems reasonable that such techniques may find their greatest utility in the solution of problems having special structures. A class of structured problems, which includes the transportation problem as a special case, is examined and it is shown that the Moore-Penrose inverse can be formed parametrically; a complete orthonormal set of eigenvectors is obtained as a set of Kronecker products. These results are then used to simplify the calculations required by certain generalized inverse methods for solving LP problems and illustrations are given using the transportation problem as a model.

TUESDAY AFTERNOON

A DECOMPOSITION APPROACH FOR SOLVING LINKED PROGRAMS

R. H. Cobb and J. Cord

The purpose of this paper is to present a simple decomposition approach for solving a linked program without destroying any original structure.

A linked program is to minimize

$$\sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n-1} d_i Y_i$$

subject to

$$\begin{aligned} A_1 X_1 + Y_1 &= b_1 & \\ -Y_{i-1} + A_i X_i + Y_i &= b_i & (i = 2, \ldots, n-1) \\ -Y_{n-1} + A_n X_n &= b_n & \\ X_i, Y_i &\geq 0 & (i = 1, \ldots, n) \end{aligned}$$

MONDAY AFTERNOON

A METHOD FOR CONVEX PROGRAMMING

M. Courtillot

Let H be a Hilbert space and K a convex set in H. It is possible to form a finite sequence $x_1, \ldots, x_n$ such that:

a) x_{k+1} is deduced from x_k by

$x_{k+1} = y_k + \rho_k (y_k - x_k) \qquad 0 \leq \rho < 1$

y_k = orthogonal projection of x_k on a plane separating x_k from K such that $d(x_k, K) \leq \varepsilon$

b) if K is nonempty, x_n belong to the neighborhood K_ε of K $(d\ (x_n, K) \leq \varepsilon)$

c) if K is empty, for any $L > 0$, there exist n such that:

$\sum_1^n (1-\rho_k)^2 \ || \ x_k - y_k \ ||^2 > L^2$.

It is possible to use this method to maximise a numerical concave function on the convex set

$K = \{x | g(x,t) \geq 0$ for all $t \varepsilon V\}$ where

V = any vector space, and $g(x,t)$ is a concave mapping of H x V in R^m, some variants are given and applications to some problems of "best approximation".

TUESDAY AFTERNOON

MINIMIZATION OF A SEPARABLE FUNCTION SUBJECT TO LINEAR CONSTRAINTS

V. De Angelis

The paper discusses a method of using separable programming to minimize non-linear functions of variables subject to linear inequality constraints. It is assumed that the objective function can be represented as the sum of non-linear functions of single arguments. Following the normal procedure in separable programming, we introduce "special variables" representing the weights attached to points on the piece-wise linear approximations to these functions. The special feature of the method is that when a special variable drops from the basis, the reduced cost of the other neighbor of the one that remains from this group is computed, and if it is negative this variable is introduced. The reason for this strategy is that the variable may well have a negative reduced cost, and its coefficients in the current tableau (assuming we are using the product form of inverse matrix method) can be computed without reference to the inverse of the basis, i.e. without either a Backward transformation or a Forward transformation. So these special iterations can be performed very quickly.

TUESDAY EVENING

LINEAR PROGRAMS WITH SEVERAL PARAMETERS IN THE OBJECTIVE FUNCTION

W. Dinkelbach

Considering any linear program it is assumed that the coefficients of the objective function depends linearly on several parameters. It will be proved that the optimal solution function, that is, the maximum of the linear program as a function of the parameters, is convex. Its minimum can be determined by a series of linear subprograms. This method will be illustrated by a numerical example.

THURSDAY AFTERNOON

CONTINUOUS MATHEMATICAL PROGRAMMING UNDER LINEAR INTEGRAL CONSTRAINTS

W. P. Drews and R. G. Segers

A dozen years ago the development of the Pontriagin Maximum Principle as a necessary condition for optimality of some control problems began a new era for optimization theory. The desirability of such a principle has motivated far-reaching generalizations which today are based on a confluence of the fields of programming, control, and functional analysis. Employing primarily the framework of mathematical programming, the present paper treats the important, but infrequently discussed, topic of optimization subject to linear integral constraints of Volterra type. The principal result is a duality-type theorem yielding a representation for the objective function, a representation valid for any trajectory resulting from an admissible control including those trajectories which are not extremals of the control problem being considered. Initially, the approach is to establish, for an arbitrary (but fixed) sufficiently smooth control, a representation of the objective function corresponding to the resulting trajectory. This then permits asserting criteria of optimality in terms of the dual or adjoint variables introduced in the functional representation.

WEDNESDAY AFTERNOON

RECENT DEVELOPMENTS IN GEOMETRIC PROGRAMMING

R. J. Duffin and E. L. Peterson

The duality theory of geometric programming, as formulated, developed, and applied by Duffin, Peterson and Zener [1], was based on abstract properties shared by certain classical inequalities, such as Cauchy's arithmetic-geometric mean inequality and Holder's inequality. By not requiring all of these properties, it is possible to generalize the duality theory to arbitrary convex programs. This generalized duality theory has symmetry not possessed by the original theory, and is closely related to (but is not identical to) the duality theory of Dorn, Huard, and Wolfe.

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WEDNESDAY AFTERNOON

MATROIDS AND EXTREMAL COMBINATORICS

J. Edmonds

Linear-algebra rank is an extremely nice extremum. We apply linear programming to it and certain generalizations.

THURSDAY AFTERNOON

THE DEGREE-CONSTRAINED SUBGRAPH PROBLEM

J. Edmonds, E. L. Johnson, and S. C. Lockhart

Previous work on maximum matching in a graph is extended to a more general degree constrained problem. The problem considered is to find non-negative integers x_k which minimize the weighted-sum $\Sigma_{k=1}^{n} c_k x_k$ subject to

$$0 \leq x_k \leq \alpha_k, \quad k = 1,\ldots,n;$$

$$\Sigma_{k=1}^{n} a_{ik} x_k = b_i, \quad i = 1,\ldots,m;$$

where $A = (a_{ik})$ is an $m \times n$ matrix with all entries +1, -1, +2, -2, or 0 and at most two non-zero entries in a column. Furthermore, if an entry a_{ik} is +2 or -2, then the kth column has no other non-zero entry. An algorithm is given for this integer program which, by contrast with other methods, has an upper bound, which is algebraic in m and n, on the amount of work and storage required. The algorithm proves that a certain class of inequalities can be adjoined to the linear program to give a convex polyhedron whose vertices are integer vectors. The problem and the algorithm are described in terms of a corresponding bi-directed graph. The inequalities are equivalent to some duality results which are interpreted in terms of the graph. A computer code has been written to implement the algorithm.

WEDNESDAY AFTERNOON

ASYMPTOTIC CONES AND DUALITY OF LINEAR RELATIONS

K. Fan

For a locally convex topological vector space E, its dual space will be denoted by E', the weak topology of E' for the dual system $\langle E, E' \rangle$ will be denoted by $\sigma(E', E)$. The polar of a set $X \subset E$ is $X^\circ = \{f \in E' : f(x) \leqq 1 \text{ for all } x \in X\}$. For a non-empty closed convex set $X \subset E$, the asymptotic cone C_X of X is $\bigcap_{\lambda > 0} \lambda(X - x)$, where x is an arbitrary point in X. The present paper uses a recent result of J. Dieudonne (Math. Annalen, 163 (1966), 1-3) to study the duality of linear relations. Two of the theorems are stated here.

Theorem A. Let E, F be two locally convex topological vector spaces. Let $A: E \to F$ be a continuous linear transformation, and ${}^tA: F' \to E'$ the transpose of A. Let P, Q be closed, locally compact, convex sets in E, F respectively such that at least one of P, Q is a cone, and

(1) $0 \in A(P) + Q$,

(2) $-x \in C_P$ and $Ax \in C_Q$ imply $x = 0$.

Then for any $y_0 \in F$, there exists an $x \in E$ satisfying $x \in P$ and $y_0 - Ax \in Q$, if $g \in Q^\circ$ and ${}^tAg \in P^\circ$ imply $g(y_0) \leqq 1$.

Theorem B. Let E, F be two locally convex topological vector spaces. Let $A: E \to F$ be a continuous linear transformation, and ${}^tA: F' \to E'$ the transpose of A. Let P, Q be closed, locally compact, convex cones in E, F respectively such that P°, Q° are locally compact in the weak topologies $\sigma(E', E)$ and $\sigma(F', F)$ respectively. Let $y_0 \in F$ and $f_0 \in E'$ satisfy the following condition:

(3) $x \in P$, $-Ax \in Q$, $g \in Q^\circ$, ${}^tAg \in P^\circ$ and $f_0(x) \leqq g(y_0)$ imply $x = 0$, $g = 0$.

Suppose that there exists an $x \in E$ satisfying

(4) $X \in P$, $y_0 - Ax \in Q$;

and that there exists $g \in F'$ satisfying

$$(5) \quad g \in Q^o, \quad {}^tAg - f_o \in P^o.$$

Then the minimum of $f_o(x)$ when x varies under condition (4), and the maximum of $g(y_o)$ when g varies under condition (5), both exist and are equal.

Even in the finite dimensional case, simple examples show that hypothesis (1), (2) in Theorem A and (3) in Theorem B are essential.

FRIDAY AFTERNOON

NONCONVEX AND CONVEX PROGRAMMING BY GENERALIZED SEQUENTIAL UNCONSTRAINED METHODS

A. V. Fiacco and G. P. McCormick

Subject to very mild conditions, the existence of a sequence of local unconstrained minima of one-parameter auxiliary or "penalty functions" is assured, such that all limit points are local solutions of a general nonconvex programming problem. The penalty functions defined are more general and the conditions weaker than have hitherto been considered. The functions define two distinct classes of sequenctial methods, those generating a feasible-interior minimizing sequence, and those generating a feasible-exterior minimizing sequence. Convergence theorems are stated and the central arguments of the proofs are summarized. It follows that the corresponding objective function values converge monotonically to a specified local minimum value. A theoretical basis is given for a procedure for accelerating convergence by extrapolation. For the convex problem, the penalty functions can be selected to be convex, local results become global, and the minimizing sequence generates a sequence of dual feasible points, limit points of which solve the dual problem.

TUESDAY AFTERNOON

MATHEMATICAL PROGRAMMING BY PHYSICAL ANALOGIES

O. I. Franksen

The purpose of the paper is to set up and discuss some analogies between, on the one hand, electrical network theory and classical mechanics and, on the other, the mathematical programming models of economics.

More specifically, an equivalent electrical network is formulated of the classical Walrasian system, the constraints of which consist of a graph combined with a set of ideal transformers representing the technical coefficients. The resulting constraint equations are found by an application of a principle of constraint-partitioning which implicitly has been given by Gabriel Kron. The elements, interconnected by these constraints, of the Walrasian system are the factor supply and demand functions. The electrical equivalents of these elements are represented by combinations of ideal voltage and current sources and, in the non-linear case, also by resistors.

One of the most surprising characteristics of the equivalent electrical network is the fact that, in general, it does not satisfy Ohm's law. Instead, therefore, it is necessary to determine the state of equilibrium by applying the principle of virtual work, in terms of Fourier's inequality, of classical mechanics to a suitably chosen state-function. The possible set of state-functions represents input or output of electric power or rate of change of energy to the network. In the non-linear case, by the Lagrangian multiplier method, these state-functions are modified by adding the electric power of the constraints.

Finally, in the discussion of the electrical network in terms of the ideas of mathematical programming, it is shown how the Kuhn-Tucker conditions are related to Kirchhoff's laws, and how the simplex method can be identified with the principle of virtual work.

WEDNESDAY AFTERNOON

THE MAX-FLOW MIN-CUT EQUALITY AND THE LENGTH-WIDTH INEQUALITY FOR REAL MATRICES

D. R. Fulkerson

The max-flow min-cut equality and the length-width inequality for two-terminal networks are extended to arbitrary real matrices. Key use is made of the frame of a subspace of Euclidean n-space, a notion closely related to that of a real matric matroid, in the generalization.

WEDNESDAY AFTERNOON

IMPLICIT ENUMERATION USING AN IMBEDDED LINEAR PROGRAM

A. M. Geoffrion

Integer programming by implicit enumeration has been the subject of several recent investigations. Computational efficiency seems to depend primarily on the ability of various tests, applied to the constraints in connection with "partial solutions," to exclude from further consideration a sufficiently large proportion of the possible solutions. Most of the simpler or more appealing of these tests can be applied at reasonable computational cost essentially to only one constraint at a time. Two main approaches have been suggested for mitigating this limitation. One is to periodically apply linear programming to continuous approximations of the subproblems generated by the partial solutions. The other approach, promulgated by Fred Glover, is to periodically introduce composite redundant constraints that tend to be useful when tests are applied to them individually. In this paper we motivate a measure of the "strength" of a composite constraint that is slightly different from the one used by Glover, and show how composite constraints that are as strong as possible in this sense can be computed by linear programming. It further develops that the dual of the required linear program coincides with the appropriate continuous approximation to the subproblems generated by the successive partial solutions. This leads to a complete synthesis of the two approaches mentioned above by means of an imbedded linear program. Computational experience is presented which confirms that this synthesis is indeed a useful one for the classes of problems tried. For numerous problems with up to 80 variables taken from the literature, the imbedded linear program typically reduced the number of required iterations by one or two orders of magnitude, and execution times by a ~~factor of between 3 and 20~~.

alter and add.

TUESDAY EVENING

DUAL CONVEX AND FRACTIONAL-CONVEX PROGRAMMING PROBLEMS

E. G. Golshtein

This report gives a general method of constructing a dual problem for a convex programming problem in a functional space.

A generalized theorem of duality is proved for an arbitrary convex programming problem. Classes of problems are specified for which the general theorem of duality is equivalent to the ordinary theorem of duality concerning the equality of extremal values of the initial and dual problems. Theorems of duality are proved in a weak or strong form (in the latter case the existence of an optimal solution of the dual problem is guaranteed). Relationships between a pair of dual problems and the problem of finding a generalized saddle point as well as between theorems of duality and criteria of optimality (in a strong or weak form). The general method of constructing a dual problem is made concrete for some particular classes of convex programming problems; the criteria of optimality are similarly made concrete. Certain properties of generalized support functionals, established in the report, are used.

A functional analogue of fractional-convex programming problem is defined in the report. A general method of constructing a dual problem for a fractional-convex programming problem is given. The results, proved for the convex case, are also proved for fractional-convex programming problems. The theory of duality developed is more general as compared with the previous approaches and leads to a number of new results even in the finite dimension case.

THURSDAY AFTERNOON

R. L. Graves

This paper presents a simplex algorithm for finding a nonnegative solution (or demonstrating the inconsistency) of $y = a + Ax$, $xy = o$ where A is positive semidefinite. Linear and quadratic programming problems are of this form. The function exhibited in the proof of finiteness does not appear in other algorithms. If a primal feasible solution is available in the linear programming case, the actual choice of pivot rows is exactly that made in the usual lexicographic simplex method.

FRIDAY AFTERNOON

ON THE KUHN-TUCKER THEORY

M. Guignard

Optimality Conditions stated below generalize the Kuhn-Tucker Conditions, while the constraint qualification is a substitute both for the Kuhn-Tucker, the Arrow, Hurwicz and Uzawa, and the Abadie constraint qualifications.

Let M be any subset in $\mathcal{R}^n$, then $\overline{M}$ and $\{M\}$ will denote respectively the closure of M and the smallest convex subset containing M.

Let $(\mathcal{R}^m)^*$ be the dual space of $\mathcal{R}^m$.

Let A be a nonempty set in $\mathcal{R}^n$, $\overline{x} \in \overline{A}$, $y \in \mathcal{R}^n$. y is a vector tangent to A at $\overline{x}$ if:

there exists a sequence (x_k) contained in A and converging to $\overline{x}$,
and there exists a sequence (ℓ_k) of nonnegative numbers,
such that the sequence $(\ell_k(x_k - \overline{x}))$ converges to y.

Let $T_A(\overline{x})$ be the set of all vectors tangent to A at $\overline{x}$, $T_A(\overline{x})$ will be called the "cone tangent to A at $\overline{x}$".

Let $P_A(\overline{x})$ be the closure of the convex hull of $T_A(\overline{x})$, i.e., $P_A(\overline{x}) = \overline{\{T_A(\overline{x})\}}$. $P_A(\overline{x})$ will be called the "pseudo-cone tangent to A at $\overline{x}$".

If Q is a cone in $\mathcal{R}^n$, let Q^- be the cone in $(\mathcal{R}^n)^*$ of all linear functionals u defined on $\mathcal{R}^n$, and such that $u \cdot y \leq 0$, for all $y \in Q$.

Let A be a subset in $\mathcal{R}^n$, let $\psi(x)$ be a real valued function in $x \in \mathcal{R}^n$. We assume that $\psi(x)$ is differentiable at $\overline{x} \in A$.

THEOREM 1: If $\overline{x}$ maximizes $\psi(x)$ subject to $x \in A$, then

$$\frac{d\psi(\overline{x})}{dx} \in (P_A(\overline{x}))^- . \tag{I}$$

Conversely, if this condition (I) is satisfied, if ψ is pseudo-concave over A, and if for all $x \in A$, $x - \overline{x} \in P_A(\overline{x})$, then $\overline{x}$ maximizes $\psi(x)$ subject to $x \in A$.

Let $a_j(x)$, $j = 1,2,\ldots,m$, be m real valued functions in $x \in \mathcal{R}^n$, and let a be (a_j), $j = 1,2,\ldots,m$. Let E be the subset of all indices effective at $\overline{x}$, and $\overline{E}$ the subset of all indices ineffective at $\overline{x}$.

$$K = \{y : \frac{da_E(\overline{x})}{dx}\ y \geq 0\}$$

Let C be a nonempty subset in $\mathcal{R}^n$, and $A = \{x \in C,\ a(x) \geq 0\}$.

THEOREM 2: (the generalized Kuhn-Tucker Conditions). If $\overline{x}$ maximizes $\psi(x)$ subject to $x \varepsilon A$, let G be a closed convex cone such that

$$K \cap G = P_A(\overline{x})$$

$$K^- + G^- \text{ is closed,}$$

then there exists $u \varepsilon (\mathcal{J}^m)^*$ such that

$$\left.\begin{aligned} &\frac{d\psi(\overline{x})}{dx} + u \cdot \frac{da(\overline{x})}{dx} \varepsilon G^-, \\ &u \cdot a(\overline{x}) = 0 , \\ &u \geq 0. \end{aligned}\right\} \quad \text{(II)}$$

Conversely, if these conditions (II) are satisfied, if A or $\{x \varepsilon {}^n : a(x) \geq 0\}$ is convex if for all $x \varepsilon A$, $x - \overline{x} \varepsilon G$, and if either ψ is pseudo-concave over A or ψ is quasi-concave and

$$\frac{d\psi(\overline{x})}{dx} \neq 0 ,$$

then $\overline{x}$ maximizes $\psi(x)$ subject to $x \varepsilon A$.

THURSDAY AFTERNOON

A FUNCTIONAL APPROACH TO THE DESIGN AND DEVELOPMENT OF A MATHEMATICAL PROGRAMMING SYSTEM

D. W. Hallene

Historically, Mathematical Programming Systems have been designed with only one objective in mind. Recent design experience has resulted in the development of a multi-objective mathematical programming system. This paper describes the results of a research and development project which produced an MPS to accomplish the following objectives: (1) Production Tool, (2) Experimental Algorithmic Development, (3) Optimization Portion of a Large Simulation System, and (4) Implementation or multiple computers. In addition, the paper discusses the basic conflict between functional modularity for ease of modification, maintenance, and extension, and the historical objective of creating the "fastest MPS to date".

Two viewpoints of the functional approach taken in the development of FMPS for UNIVAC 1108, IBM S/360, and RCA Spectra 70 computers are discussed. The first viewpoint is the impact of the various operating systems and hardware on data storage structure and functional file organization. The second viewpoint is the algorithmic concepts which must be considered when developing Linear Programming, Decomposition, and Generalized Upper Bound as an integrated system.

MONDAY EVENING

A HYPERCONE SEARCH OPTIMISATION METHOD

L. Haller

Derivatives are not employed. The procedure uses relatively few function values to minimize a function of several variables with prescribed "accuracy". Quadratic iterative optimization along a direction with step-size adaptation. Convergence criteria. Direction averaging. Adaptive hypercone of new admissible directions. Stopping rules. Computed examples and comparisons. Noisy data minimization. Reduction of constrained problems to unconstrained minimization; difficulties and results attained. Discussion of limitations on the type of problem.

TUESDAY EVENING

MaGEN II

C. A. Haverly

Matrix and Report Generation has become a major part of modern mathematical programming systems. This paper discusses MaGen II, a high level language developed specifically for this purpose.

MaGen II is the result of many man years of development. It is currently implemented and operational on the IBM 360 series of computers as well as the Honeywell 200 and IBM 1400 series. The details of the language, the rationale behind the various features, and a report of user experience will be given.

The approach used in MaGen is based on a recognition that mathematical models consist of activities and constraints on these activities, and that both the activities and constraints can be grouped into classes. The generation of the matrix is carried out by FORM VECTOR statements under control of a DICTIONARY which defines the classes and provides mnemonic names for use in the model, and a Data section which provides the numerical information.

The report desired from a mathematical programming solution can be structured in terms of classes of lines. These are generated be FORM LINE statements under control of the DICTIONARY and Data.

Powerful data and file manipulation capabilities are included to handle the large-complex modeling situations. MaGen II has been applied in a wide variety of models including multi-product, multi-plant, multi-time period models.

MONDAY EVENING

A PARTICULAR CLASS OF FINITE-TIME MARKOV-RENEWAL PROGRAMMING SITUATIONS

J. R. Hemsley

A class of situations are described in which decisions have to be made sequentially over a finite time period. Times between events (transition times) are distributed according to probability distributions which depend only upon the states between which the transitions are made. The characteristic property of the processes studied in this paper is that of continuous depletion (or accretion) throughout the time period, i.e. all transition probability matrices are triangular as found in certain problems in Stock Control and Dam Theory. This property enables the optimal policy to be determined by the use of particular embedded Markov Chains without the necessity of solving integral equations at each stage of the policy determination process as is found in the usual Markov Renewal Programming Problem over a finite time period.

TUESDAY EVENING

WHEN IS A TEAM "MATHEMATICALLY" ELIMINATED?

A. J. Hoffman and T. J. Rivlin

Our purpose is to give a different proof of a result of B. L. Schwartz ("Possible Winners in Partially Completed Tournaments", SIAM Review, vol. 8, 1966, pp. 302-308.) and generalize it somewhat.

The setting for our problem is a league of n teams. In the course of a season each team plays m games with every other team. Each game results in one of the contesting teams winning and the other losing. At the end of the season the teams are arranged, in inverse order, according to the number of games they have won, in places $1,\dots,n$, and the team (or teams) which has won the most games wins (or ties for) the pennant. We wish to determine the highest place to which a team (or set of teams) may aspire at a given time in the course of the season, knowing how many games each team has won up to that time, and how many games remain to be played in the season between each pair of teams.

At present, in the case of either of the baseball major leagues in the United States, $n = 10$ and $m = 18$. (Games which end in a tie are generally replayed at later dates and a rule making this mandatory will probably be adopted soon. We shall assume that this rule obtains). There are two simple criteria in popular use to determine when a team is "mathematically eliminated" in the pennant race: i) If our team were to win all of its remaining games and its total of games won for the season would then be less than the number of games presently won by another team (usually the team currently leading the league), then our team is eliminated. ii) If our team has lost 82 games (one more than half the total number of games each team plays in a season) it is eliminated since a moment's reflection reveals that some team in the league must end the season with at most 81 games lost. We shall obtain Schwartz's necessary and sufficient conditions that a team be eliminated from pennant contention and thereby show that the traditional journalistic criteria may sometimes be improved upon. Additionally, we find necessary and sufficient conditions for a team to be eliminated from finishing in t^{th} place, $t > 1$, and for a given set of k teams to aspire to fill the first k positions at the end of the season.

THURSDAY AFTERNOON

NONLINEAR PROGRAMMING AND SECOND-VARIATION SCHEMES IN CONSTRAINED OPTIMAL CONTROL PROBLEMS

G. Horne and G. S. Tracz

Formal interest in the application of mathematical programming techniques to optimal control problems culminated in the organization of the First International Conference on Programming and Control, April 15-16, 1965. For example, in one of the papers, Dantzig studied the application of the decomposition principle in the form of the generalized linear program to a class of linear control processes.

A large class of nonlinear optimal control problems can be formulated in the following manner:

"Minimize the scalar objective function J,

$$J = \int_{t_0}^{t_1} L[x(t),u(t),t]dt \tag{1}$$

where $x(t)$, an $(n\times 1)$ vector of time functions called <u>state</u> variables, is defined by

$$\dot{x}(t) = f[x(t),u(t),t], \quad x(t_0) = a \tag{2}$$

and $u(t)$, a $(m\times 1)$ vector of functions called <u>control</u> variables, is constrained as follows,

$$|u(t)| \leq M." \tag{3}$$

Proposed gradient (that is, first-order) schemes for choosing the optimal control vector $u(t)$ are numerous, but the resulting rate of convergence in the vicinity of the optimum solution has been found to be generally slow. Second-order schemes, more commonly known as second-variation schemes, are more efficient in the neighbourhood of the optimum but they have been developed so far only for the unconstrained control problem--that is, with condition (3) removed.

The purpose of this paper is to present a second-variation scheme that includes (3). The procedure is iterative in nature. At each iteration, a quadratic approximation $\delta\bar{J}(\delta u)$ to the change in an augmented objective function $\bar{J}$ is obtained where

$$\delta\bar{J}(\delta u) = \int_{t_0}^{t_1} \frac{\partial H}{\partial u}\,\delta u(t)dt + \frac{1}{2}\int_{t_0}^{t_1}\int_{t_0}^{t_1} \delta u^T(t)W(t,s)\delta u(s)ds\,dt \tag{4}$$

subject to

$$|\delta u(t)| \leq \xi \tag{5}$$

The functions $\delta u(t)$ are allowable pertubations in the control vector $u(t)$. The matrix $W(t,s)$ is called a second-order weighting matrix. H is the Hamiltonian function

$$H(x,p,u,t) = L(x,u,t) + p^T(t)\ f(x,u,t), \tag{6}$$

where $p(t)$ is a vector of unknown Lagrange multiplier functions, related to H by,

$$\dot{p}(t) = -\frac{\partial H}{\partial x}, \quad p(t_1) = 0 \tag{7}$$

and $(\ )^T$ denotes the transpose of $(\ )$.

Thus the original control problem has been transformed into that of maximizing the improvement $\delta \overline{J}(\delta u)$, expressed in the form of a quadratic functional, subject to constraints (5). This recast problem can be considered as an infinite-dimensional nonlinear programming problem. The generalized Kuhn-Tucker theorem in nonlinear programming is then used to derive a system of equations which define the resulting form for the control law $\delta u(t)$.

Other aspects of this scheme are also discussed--for example, the duality problem. In addition, it should be noted that the problem of maximizing the expression given in (4) subject to constraint (5) also falls directly into the domain of quadratic programming problems.

Numerical examples for both nonlinear and linear problems with a single control variable are included. The resulting convergence rate with the proposed scheme in this paper has been found to be satisfactory.

WEDNESDAY AFTERNOON

A DECOMPOSITION ALGORITHM FOR SHORTEST PATHS IN A NETWORK

T. C. Hu

Given a network with distances defined on the arcs, the problem is to find shortest paths between every pair of nodes in the network. If the network can be disconnected by removing a subset of the nodes, then it is possible to treat parts of the network at a time to save the amount of computation as well as shortage requirements.

WEDNESDAY AFTERNOON

PROGRAMMES MATHÉMATIQUES NON LINÉAIRES À VARIABLES BIVALENTES

P. Huard

Nous nous proposons de décrire un algorithme heuristique, qui s'est révélé efficace, permettant de résoudre des Programmes Mathématiques, linéaires ou non, à variables bivalentes, soit:

Maximiser $\psi(x)$ sous les conditions

(1) $a_\ell(x) \geq 0, \quad \ell \in L = \{1,2,\ldots,m\}$

(2) $x_j = 0$ ou $1, j \in J = \{1,2,\dots,n\}$

ou ψ et les a_ℓ sont des fonctions numériques.

Dans le cas où les fonctions sont linéaires, les calculs sont plus rapides, car on est ramené à une séquence finie de Programmes linéaires en variables continues, de taille équivalente au Programme donné. Mais dans son principe, la méthode est la même pour les cas linéaires ou non linéaires. Elle a pu d'autre part être étendue directement avec succès à des Programmes non linéaires mixtes (c'est-à-dire dont certaines variables sont bivalentes et les autres continues).

PRINCIPE DE LA METHODE

Chaque fois que l'on détermine un sommet $\overset{k}{x}$ de l'hypercube unité vérifiant les contraintes (1) (sommet réalisable), on tronque le domaine des solutions réalisables par la contrainte supplémentaire $\psi(x) \geq \psi(\overset{k}{x})$. Soit T_k ce troncon, qui contient les solutions réalisables continues "meilleures" que $\overset{k}{x}$.

On détermine un nouveau sommet réalisable de l'hypercube unité, soit $\overset{k+1}{x}$, appartenant à T_k, par le procédé suivant: après avoir déterminé un centre c de T_k, au sens défini dans [1], on dresse la liste des sommets x de l'hypercube dans l'ordre des distances (euclidiennes) $d(x,c)$ croissantes.

Au cours du déroulement de cette liste, à chaque nouveau sommet appelé, on vérifie s'il satisfait aux contraintes (1) - Si non, on appelle le suivant, si oui, ce sommet est le point $\overset{k+1}{x}$ cherché.

On tronque alors T_k par $\psi(x) \geq \psi(\overset{k+1}{x})$ et on recommence.

La méthode converge en un nombre fini de troncatures (pratiquement 4 à 6). Les sommets réalisables ont toujours été bien groupés au début de la liste, et leur recherche est rapide (additions et tests). La détermination du centre d'un troncon se ramène à la maximisation approximative d'une fonction sans contraintes dans le cas non linéaire, et à un Programme linéaire à variables continues dans le cas linèaire.

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REFERENCE:

[1] P. Huard, "Résolution des P.M. à contraintes non linéaires par la Méthode des Centres". Note E.D.F. N° HR 5690 du 6.5.64 - Version anglaise dans "Nonlinear Programming", Ed. J.Abadie, North Holland Publishing Co., Amsterdam, 1967.

THE CURRENT STATE OF CHANCE-CONSTRAINED PROGRAMMING

M. J. L. Kirby

This paper presents a survey of recent developments in chance-constrained programming. The motivation for a chance-constrained formulation of a "real world" problem is discussed through reference to recent applications of the technique to problems in transportation, finance and media

selection. The methodology which has been used to develop properties of optimal decision rules is analysed and the main results summarized. In addition, the notion of a chance-constrained game and properties of such games are discussed.

RECENT RESULTS ON THE COMBINATORIAL STRUCTURE OF CONVEX POLYTOPES

V. Klee

In recent years the study of the combinatorial (facial) structure of convex polytopes has been greatly stimulated by its connections with linear programming. This lecture will survey recent developments which are related to linear programming and to integer programming.

MATHEMATICAL PROGRAMMING AND PROJECT INTERRELATIONSHIPS IN CAPITAL BUDGETING

A. K. Klevorick

The usefulness of mathematical programming methods in analyzing capital-budgeting problems is, by this time, widely recognized. The research of which the present paper forms a part applies programming methods to the study of capital budgeting in the presence of risk and capital-market imperfections. In this paper, one particular aspect of the programming model developed in the larger study is emphasized: the presence of direct cash-flow interrelationships among the investment projects.

Budget constraints in several periods and limitations on the amounts that can be borrowed during periods within the firm's horizon force the fate of one proposal to depend on the firm's decisions concerning other possible projects. Internal and external restrictions on funds thus induce *indirect* interdependences among projects. In addition, physical conditions of mutual exclusion or contingency constitute a set of *direct* interrelationships among projects. The interactions on which the present paper focuses are, in contrast, the *direct cash-flow* interactions, both in outlays and returns, that exist among projects apart from the presence of financial constraints.

The sources of these interactions are discussed. It is indicated how both technological elements and demand patterns give rise to such interrelationships. The importance of considering deterministic cash-flow interactions if the firm is to reach an optimum is readily made clear. When one turns to the risk environment, the case for considering stochastic cash-flow interactions is seen to be equally compelling. In the latter case, attitudes toward risk impose certain requirements on the form of the objective function, in addition to the restrictions that were present in the certainty model. These requirements definitely necessitate consideration of stochastic cash interrelationships among projects.

The resulting programming model of capital budgeting under risk when specific capital-market imperfections exist is shown to be a mixed-integer programming problem with nonlinearities in the integer parts of the objective function and constraints. The nonlinearities are due to the cash-flow interactions among projects. By introducing additional constraints and artificial variables, there exist several ways to convert the problem into a mixed-integer programming problem with linear constraints and linear objective function. Considerations of usefulness for economic analysis and computational considerations that enter into making a choice among these alternative methods are discussed.

One method is selected on the basis of these considerations, although the choice is made, unfortunately, without the aid of computational experience. Benders' partitioning procedure is used to solve the problem which now has a linearized objective function and constraint set. An interesting economic interpretation can be lent to the algorithm's progress toward a solution. The paper closes with a discussion of the decision mechanism represented by the solution procedure.

THURSDAY AFTERNOON

OPTIMUM DESIGN OF LONG PIPE LINE NETWORKS FOR GASES USING LINEAR PROGRAMMING

J. P. Kohli and F. W. Leavitt

A FORTRAN program has been written for the IBM system/360 to design gas distribution piping networks using linear programming. Such networks are directed trees. Gas is supplied at a specified supply pressure at the root of the tree and is drawn off at the branches at specified delivery rates and at specified minimum delivery pressures. The program sizes pipe in all parts of the net to minimize installed cost subject to flow rate and delivery pressure constraints. Standard pipe sizes are used everywhere. Each link in the net may contain more than one pipe size. The Weymouth formula is used to calculate pressure drops with the acceleration term neglected. Pressure gradient is a function of pressure but a change in variable has been used to allow use of linear programming.

THURSDAY AFTERNOON

CENTRALIZATION AND DECENTRALIZATION OF DECISION MAKING

THE DOUBLE DECOMPOSITION METHOD

GENERALIZATION AND PROOF OF CONVERGENCE

T. O. M. Kronsjo

A proof is given for that the double decomposition method proposed by D. Pigot may be generalized to deal with any type of linear programming problem and that, if certain conditions are observed, the method will converge in a finite number of iterations to the optimal solution. As for certain purposes an economic system may be approximated in the form of a linear programming problem, this decomposition procedure is of profound theoretical and practical importance in indicating a possible system for optimal planning based upon a combination of central price and quantity parameters (cp. the decomposition methods of Dantzig-Wolfe and Benders).

The approach is based upon the following results.

1. A linear programme

$$\underset{x}{\text{Min}} \left\{ cx \mid Ax \geq b, \quad x \geq 0 \right\},$$

no matter whether it has some type of solution or not, may be evaluated by the solution of two or three related linear programmes with finite optimal primal and dual solutions, viz.,

$$\underset{x,y}{\text{Min}} \left\{ 0x + 1y \mid Ax + y \geq b, \quad x \geq 0, \quad y \geq 0 \right\},$$

$$\underset{u,v}{\text{Max}} \left\{ u0 - v1 \mid uA - v \leq c, \quad u \geq 0, \quad v \geq 0 \right\}, \quad \text{and}$$

if the above problems both equal zero then

$$\underset{x}{\text{Min}} \left\{ cx \mid Ax \geq b, \quad x \geq 0 \right\}.$$

2. A linear programme with optimal finite primal and dual solutions, say the immediately preceding problem, may be extended into an equivalent enlarged problem, say,

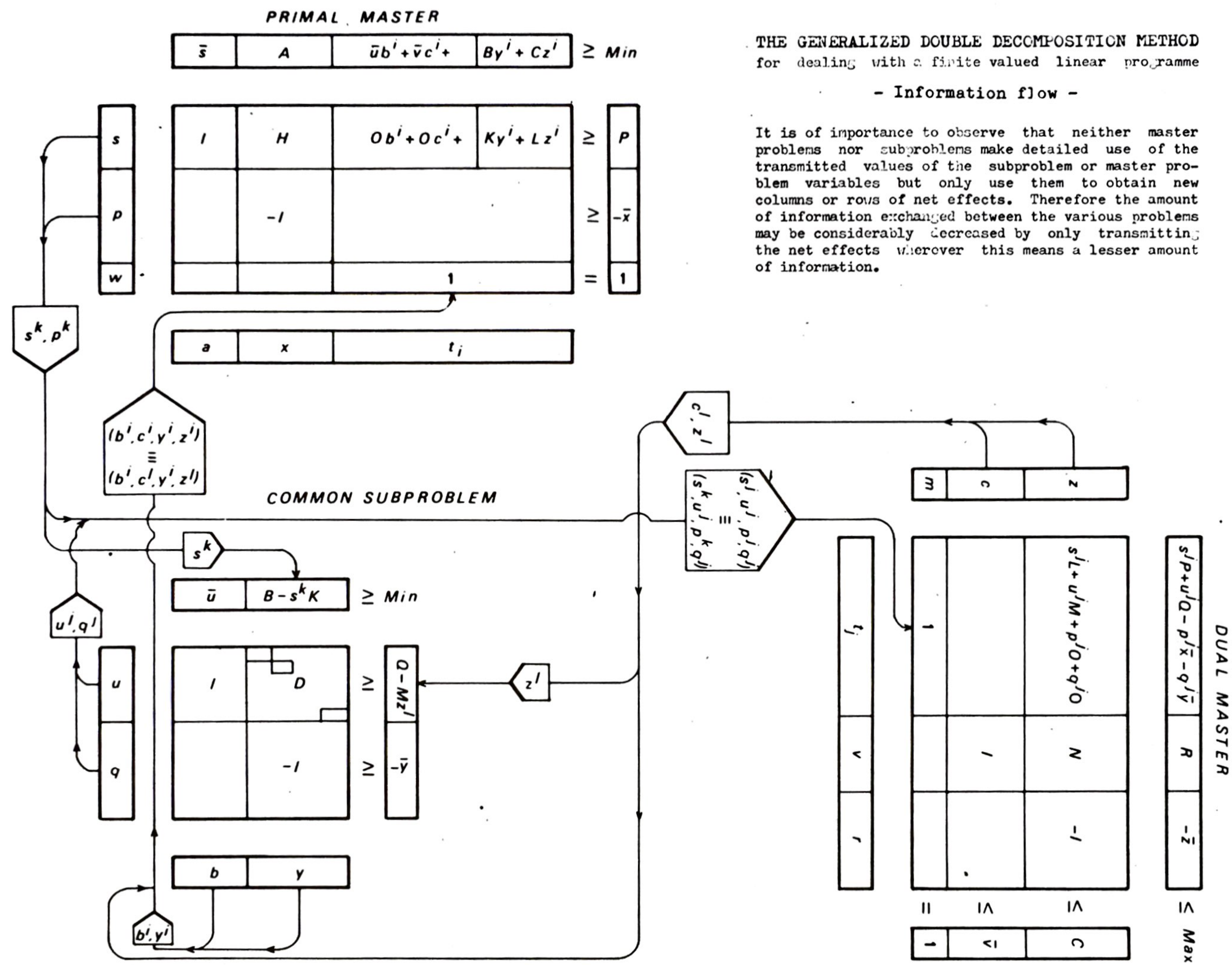
$$\underset{x,y}{\text{Min}} \left\{ \bar{u}y + cx \mid y + Ax \geq b, \ -x \geq -\bar{x}, \quad y \geq 0, \quad x \geq 0 \right\},$$

to which initial feasible primal and dual solutions may easily be found.

3. A linear programme with finite optimal primal and dual solutions to which initial feasible primal and dual solutions are known

$$\underset{a,b,c,x,y,z}{\text{Min}} \left\{ \begin{array}{cccccccccccl} \bar{s}a & + & \bar{u}b & + & \bar{v}c & + & Ax & + & By & + & Cz & \Big| \\ a & & & & & + & Hx & + & Ky & + & Lz & \geq P \\ & & b & & & & & + & Dy & + & Mz & \geq Q \\ & & & & c & & & & & + & Nz & \geq R \\ & & & & & & -x & & & & & \geq -\bar{x} \\ & & & & & & & & -y & & & \geq -\bar{y} \\ & & & & & & & & & & -z & \geq -\bar{z} \\ \multicolumn{12}{c}{a \geq 0 \quad b \geq 0 \quad c \geq 0 \quad x \geq 0 \quad y \geq 0 \quad z \geq 0} \end{array} \right\}$$

may be solved in a finite number of iterations by decomposing it into primal and dual directions and obtaining a primal master, a dual master and a common subproblem as indicated in the following figure.



THE GENERALIZED DOUBLE DECOMPOSITION METHOD for dealing with a finite valued linear programme

- Information flow -

It is of importance to observe that neither master problems nor subproblems make detailed use of the transmitted values of the subproblem or master problem variables but only use them to obtain new columns or rows of net effects. Therefore the amount of information exchanged between the various problems may be considerably decreased by only transmitting the net effects wherever this means a lesser amount of information.

The conditions for transfer from primal to dual iterations (respectively vice versa) may be summarized as

i) no one of the a,x(v,r) and slack variables may be a candidate for introduction into the basis of the primal (dual) master; and

ii) the value of the primal (dual) master plus the corresponding minimal (maximal) value of the primal (dual) subproblem using the simplex multipliers of the dual (primal) master must be greater (less) than the value of the dual (primal) master in order that the solution of the latter may be improved by the immediate introduction of some sufficiently optimal dual (primal) subproblem solution based upon the last simplex multipliers of the primal (dual) master.

The favourable computational experiences of Beale, Small and Hughes with their large primal decomposition programme may probably be taken as an indication that the above generalization of the double decomposition method may offer not only an important theoretical but also a forceful practical tool for achieving a near optimal solution of very large economic planning systems, especially for optimal international/interregional/interenterprise economic planning.

A great advantage of the method is that the solution of a common subproblem of blockdiagonal structure may take place in parallel, each block independently of the other. A very large economic planning problem containing a reasonable number of joining resource balances and joining activities and a great number N of diagonal blocks each of reasonable size, may therefore, in principle, be solved on N computers each solving one block of the common subproblem plus 2 computers solving the primal and dual master problems.

MONDAY AFTERNOON

NONLINEAR PROGRAMMING AND ENGINEERING DESIGN

L. S. Lasdon and A. D. Waren

The performance specifications for many engineering design problems can be formulated as a system of inequalities. Usually the designer selects a structure for the system and thus reduces the design problem to the determination of a parameter vector. Often there are additional inequality constraints on these parameters, imposed by realizability and economic considerations. An acceptable design is then a parameter vector which satisfies both these sets of inequalities.

By introducing an additional independent variable, ξ, the problem of finding an optimal design can be formulated as a mathematical program. Its solution is optimal in the following sense: if $\min \xi < 0$, the design maximizes the minimum amount by which the performance specifications are met while if $\min \xi \geq 0$, the design minimizes the maximum amount by which these specifications are not met. Solution of the mathematical program is accomplished by the SUMT technique of Fiacco and McCormick, with the unconstrained sub-problems solved using the Fletcher-Powell descent method.

This approach has been applied to the design of electrical filter networks with bounded lossy elements in both the frequency and time domain and to the design of linear and planar sonar transducer arrays. Let the response function for the system be $F(\underline{e}, \alpha)$ where $\underline{e}$ is the parameter vector and α may be either frequency, time, or a position co-ordinate.

For frequency domain filter design F could be the insertion loss and α the frequency, for time domain networks F represents the step or impulse response and α the time, while for transducer arrays F is the array directivity as a function of the angle α.

As an example consider the filter design problem. Here the performance specifications require the insertion loss to exceed or equal given values within stop band regions and to be less than or equal to given levels in a passband region. This leads to the following system of inequalities

$$F_i(\underline{e}) \geq \beta_i \qquad i = 1, \ldots, a$$

$$F_i(\underline{e}) \leq \beta_i \qquad i = a+1, \ldots, b$$

$$F_i(\underline{e}) \geq \beta_i \qquad i = b+1, \ldots, c$$

and

$$\ell_i \leq e_i \leq u_i \qquad i=1, \ldots, n$$

where

$$F_i(\underline{e}) \equiv F(\underline{e}\ \alpha_i),$$

the α_i are appropriately chosen frequencies, and ℓ_i and u_i are the lower and upper bounds on the i^{th} network element.

The solution of the following nonlinear program represents an optimal design:

$$\text{minimize } \xi$$

subject to

$$F_i(\underline{e}) + \xi \geq \beta_i \qquad i = 1, \ldots, a$$

$$F_i(\underline{e}) - \xi \leq \beta_i \qquad i = a+1, \ldots, b$$

$$F_i(\underline{e}) > \xi \geq \beta_i \qquad i = b+1, \ldots, c$$

and

$$\ell_i \leq e_i \leq u_i \qquad i = 1, \ldots, n$$

Despite the non-convexity of this nonlinear program, computational experience has been very satisfactory. A number of filters have been designed which are superior to those previously available. Similar results have been obtained for the other problems considered. In one planar array design, the problem involved specifications on five separate, simultaneously generated directivity patterns at ten angles in each of three planes leading to a nonlinear program with 71 variables and 150 nonlinear inequalities.

THURSDAY AFTERNOON

NONLINEAR PROGRAMMING VIA ROTATIONAL DISCRIMINATION

V. J. Law and R. H. Fariss

A method is developed for solving the general nonlinear programming problem. The method is based upon the principles of the rotational discrimination method of Fariss and Law [1] for solving nonlinear regression problems. Some of the more important features of the proposed algorithm are as follows:

1) Parallel convergence is achieved. That is, moves are made from a base point which seek to satisfy the constraints while simultaneously working toward an optimum. Thus, intermediate feasibility is not required.

2) The search direction is chosen by first dealing with the constraints and then with the objective function. This has the significant effect of <u>reducing the dimensionality</u> of the optimization portion of the problem.

3) The search trajectory has "truncation convergence." That is, progress toward a solution is guaranteed.

4) The definition of a "beneficial objective function" allows proper monitoring of progress along the search trajectory.

5) Quadratic convergence is available if the user is willing to allow the required second derivative calculations. However, the method is flexible in that this portion may be retained or deleted at will.

TUESDAY EVENING

COMPLEMENTARY SOLUTIONS FOR SYSTEMS OF LINEAR EQUATIONS

C. E. Lemke and R. W. Cottle

Given the square matrix M and column q, a non-negative solution to the system of n linear equations in 2n variables

$$w = q + Mz$$

which satisfies

$$z_i w_i = 0 \qquad \text{for } i = 1,\ldots,n$$

is a <u>complementary solution</u> for the system.

Various classes of matrices M are studied for which statements of existence of complementary solutions can be made, and characteristic properties of these classes are identified. The question of computing complementary solutions is discussed.

FRIDAY AFTERNOON

OPTIMALITY AND DUALITY IN NONLINEAR PROGRAMMING IN THE PRESENCE OF EQUALITY CONSTRAINTS AND AN APPLICATION IN NONLINEAR DISCRETE OPTIMAL CONTROL

O. L. Mangasarian

Almost all sufficient optimality criteria and duality relations of nonlinear programming treat only linear equalities. By observing that a function can be both quasi-convex and quasi-concave without being linear, sufficient optimality conditions and duality results are obtained for problems with special types of nonlinear equality constraints. Similarly, by observing that a function may be both pseudo-convex and pseudo-concave without being linear, necessary optimality conditions are obtained for problems with certain nonlinear equality constraints. As an application of these results we obtain sufficient optimality criteria, necessary optimality criteria, and duality relations for a class of nonlinear discrete optimal control problems.

TUESDAY EVENING

ON THE EXISTENCE OF OPTIMAL PROGRAMS OF ECONOMIC GROWTH IN INFINITE-HORIZON LINEAR MODELS

D. McFadden

Three problems in the programming of economic growth in abstract linear economies are solved in this paper.

(1) A class of partial orderings of outcomes are shown to yield well-defined optima in cases where conventional objective functionals do not exist.

(2) An algebraic extension of the Kuhn-Tucker theorem is established for a class of problems in which a monotone partial ordering is maximized over the non-negative orthant of a denumerably infinite-dimensional vector space. Conditions are found under which this result implies dual prices at which an optimal program can be sustained in a decentralized economy.

(3) A uniform boundedness condition on instantaneous objective functions is shown to be a necessary and sufficient condition for the existence of an optimal program in a class of stationary economies.

This paper extends the results contained in my article "The Evaluation of Development Programs" Review of Economic Studies, Jan. 1967, and covers material from two unpublished manuscripts, "On Malinvaud Prices", W.P. No. 123, Center for Research in Management Science, Berkeley, 1965, and "The Evaluation of Development Programs: Addenda," (mimeographed), University of Chicago, April 1967.

THURSDAY AFTERNOON

ON THE CAPACITY OF A DISCRETE, CONSTANT CHANNEL

B. Meister and W. Oettli

The determination of the capacity of a discrete noisy channel in the case of unequal duration of signals amounts to a programming problem:

$$(1) \quad \max_{z\varepsilon Z} T(z)$$

where

$$Z = (x,y), \quad T(z) = \frac{\sum_j a_j x_j - \sum_i y_i \log y_i}{\sum_i t_j x_j},$$

$$Z = \{z \mid x \geq 0, \ \sum_j x_j = 1, \quad y = Px\},$$

$t_j > 0$, P a stochastic transition matrix.

More generally we consider the following problem:

$$(2) \quad \max_{z\varepsilon Z} \frac{f(z)}{g(z)},$$

where $f(z)$ concave and >0 over Z,

$g(z)$ convex over Z, and

$g(z^0) + (z-z^0)^T \nabla g(z^0) > 0$ for all $z^0 \varepsilon Z$, $z \varepsilon Z$.

This is an example of a quasiconcave programming problem (even pseudoconcave in the terminology of Mangasarian) which can be completely solved both theoretically and computationally. We give necessary and sufficient conditions for optimality and a procedure for finding an optimal solution, involving the fractional linearization

$$\tau_{z^0}(z) = \frac{f(z^0) + (z-z^0)^T \nabla f(z^0)}{g(z^0) + (z-z^0)^T \nabla g(z^0)}$$

of $T(z)$ at various prints $z^0 \varepsilon Z$. The procedure has the convenient feature of furnishing convergent lower and upper bounds for the optimal value. For the original problem (1) we also derive from our optimality conditions a dual problem, which has been stated earlier by E. Eisenberg for the case $t_j = 1$, all j. This is done in an elementary way, without using Lagrange-multipliers.

THURSDAY AFTERNOON

Amy Thomas Get

EXTENDING NEWTON'S METHOD TO SYSTEMS OF INEQUALITIES

H. D. Mills

Newton's method for solving systems of equations is extended to systems of inequalities which are characteristic of constrained optimization and game equilibrium problems defined by differentiable payoff functions. Local convergence at solutions is quadratic, though global ambiguities of Newton's method are still inherited. Numerical experience will be described.

MONDAY AFTERNOON

A MATRIX GENERATOR AND REPORT WRITER SYSTEM FOR USE WITH 360 MPS

M. Minns

This paper describes a matrix generator and report writer system which has been written in FORTRAN for use with 360 MPS. It incorporates all the usual features to be found in such systems, for example, comprehensive data checks including a clear language print-out of the input data, facilities for the selection of sub-models, complete control over the size, style and content of the solution report, etc.

The system was initially designed as a tool for long term planning, in particular project evaluation, in interrelated chemical complexes in Imperial Chemical Industries (Mond Division). However, its potential use for other mathematical programming applications within the Division was quickly realised and the system now forms part of a system for short-term planning in a particular chemical complex. The mathematical programming problem associated with this complex is non-linear because many of the product flows are multi-component ones and have alternative routes. Since, at present, there is no non-linear algorithm in 360 MPS the problem has been formulated with linear approximations to the non-linear constraints. The problem can then be solved using 360 MPS and in the light of the solution the approximations can be revised and the problem resolved. This process is repeated until convergence to within a specified tolerance has been achieved.

The basic system also forms part of a weekly scheduling system which is used to produce the optimum production and distribution schedule for a multi-product complex taking into account both customer demands and subjective stock requirements.

Each of these three main areas of application is described in detail and indications are given as to the developments which can be expected in these areas in the future.

MONDAY EVENING

ON THE COMPLEXITY OF DISCRETE PROGRAMMING PROBLEMS

J. Morávek

The paper deals with the estimations of the number of ordering relations ">, =, <" needed for solving of a discrete programming problem. There is given a general definition of the algorithm for solving the discrete programming problem. A great number of well-known algorithms may be described by means of this definition. The paper also includes non-programming applications of the introduced concepts. At whole, the paper may be considered as a contribution to the general theory of Discrete Programming.

TUESDAY EVENING

ILL-CONDITIONING OCCURRING IN PENALTY AND BARRIER FUNCTIONS

W. Murray

The problem of

$$\min \ (F(x))$$

subject to

$$G_j(x) \geq 0 \qquad j = 1,\ldots,m$$

is related to that of minimizing the sequence of functions

$$\text{a)} \quad P(x,r_k) = F(x) + r_k^{-1} \sum_{j=1}^{m} \delta_j \, \phi(G_j(x)) \qquad k = 0, 1, 2,\ldots.$$

$$\text{where} \quad \delta_j = 0 \quad G_j(x) \geq 0 \qquad j = 1, 2,\ldots,m$$
$$= 1 \quad G_j(x) < 0$$

$$\text{b)} \quad B(x,r_k) = F(x) + r_k \sum_{j=1}^{m} \theta(G_j(x)) \qquad k = 0, 1, 2,\ldots.$$

Functions of type (a) we refer to as Penalty functions and those of type (b) as Barrier functions. The problem of minimizing the sequence (a) or (b) is an ill-conditioned one. The nature and contrasts of this ill-conditioning is discussed and also the effect this has on the algorithms required to solve the problem.

THURSDAY AFTERNOON

THE SHIFTING-OBJECTIVE ALGORITHM: AN ALTERNATIVE METHOD FOR SOLVING GENERAL LINEAR PROGRAMMING PROBLEMS

V. Nalbandian

The simplex method has long been a generally used technique for solving linear programming problems. An alternative algorithm, believed to be more efficient, is the subject matter of this study. The major contributions of the present study are a proposed new method for solving general linear programming problems and a statistical analysis for comparing the new algorithm with the simplex method. In addition, a number of related issues such as finding an initial basic feasible solution, resolving degeneracy, obtaining the values of the dual variables, etc., are discussed.

Computational experience has been gained by solving a set of 61 test problems with randomly generated coefficients by means of the proposed algorithm as well as the standard simplex method. All of the problems solved are initially primal feasible. The number of constraints ranges from 50 to 100.

The proposed algorithm has required a total of 3,420 pivots to solve the set of 61 test problems as opposed to 4,301 pivots required by the standard simplex method. This represents a 20.5 per cent savings in the total number of pivots. It is argued, in the paper, that the amount of computation per iteration required by the proposed algorithm is insignificantly greater than that required by the standard simplex method. The statistical analysis, based upon the results obtained by solving the set of 61 test problems, suggests that the performance of the proposed algorithm, relative to the simplex method, is likely to improve as the size of the matrix of constraint coefficients increases and as the percentage of negative coefficients (after making all elements of the requirements vector nonnegative) increases. The peculiarities of the proposed algorithm, as compared to the simplex method, are:

1. There is no permanent objective row. After each pivot, a new row is selected to serve as the current objective row. The selection process involves a small amount of computation and, barring degeneracy, the objective row is uniquely determined after each pivot. An objective row can never succeed itself, although the same row may serve as an objective row several times throughout the iterative process. This is the reason why the proposed algorithm is referred to as the shifting-objective algorithm.

2. A coefficient in the current objective row may be selected as the next pivot element.

3. After a pivot column has been selected, the pivot row may be chosen either without any further computation or by solving an auxiliary linear programming problem involving only two nonbasic variables. The former procedure gives rise to the naive shifting-objective algorithm and the latter to the modified shifting-objective algorithm.

4. The objective variable is nonbasic except initially. Unlike other nonbasic variables, its value is not assigned a zero value.

5. The requirements vector does not remain nonnegative. Feasibility is nevertheless maintained by assigning an appropriate value to the objective variable. As a result, the value of each basic variable becomes a linear function of the objective variable.

THE BLOCK PRODUCT DECOMPOSITION ALGORITHM

W. Orchard-Hays

The concepts of decomposition of an LP model, computing with partitioned blocks, and maintaining a structured basis inverse, are all quite old. The author worked with Dantzig on block triangular algorithms as early as 1953-54. However, neither computer technology nor algorithmic maturity were equal to the task at that time. Decomposition received wide interest only after publication of the Dantzig-Wolfe (D-W) algorithm in 1959.

The particular form of model used by D-W--a segmented master problem with diagonal blocks below the segments--has become a standard form of decomposition model. The D-W method is relatively easy to prove and explain and can be generalized in several ways. However, it has computational inefficiencies and judgment is often required for solution strategy, thus making automation difficult. Other algorithms of greater complexity and efficiency have been devised but they lack the elegance of simplicity of concept and generalization of subproblems which is inherent in D-W.

Nevertheless, for strictly linear problems of the D-W structure, a highly efficient and automated algorithm is feasible. The algorithm presented is a generalization of the product form of inverse to blocks. This maintains a pseudo basis inverse which is corrected to the true inverse with an additional blocked transformation matrix T, similar to the early block triangular algorithms. However, it will be shown that the extent of the non-trivial part of T can be held to no more columns that the number of rows in the master problem, and that none of these columns is longer than the number of rows in the largest subproblem.

The algorithm is based on the parametric right hand side technique. An optimal solution to a modified set of master problem constants is first obtained in a straightforward manner by solving the single master block and each master-subpartition in succession. The right hand side is then parameterized to its original value.

Parametric right hand side and objective function algorithms are merely extensions to the method and, hence, are applicable to decomposition models.

A simplified procedure has been devised for taking advantage of a previous solution when continuing a problem or solving a modified model. Infeasibility of any subproblem is quickly detected as in D-W but infeasibility of the entire problem is displayed by a premature maximum of the parameter. The resulting solution is optimal and feasible for a modified problem which is the "closest" in some sense to the original.

The problem of unbounded subproblems is handled with a unique type of mixed convexity restraint which depends on logic instead of "large" numbers.

The algorithm is currently in checkout on the GE 635 (LP/600) and is being programmed for the CDC 6600 Optima MPS.

ON PROBABILISTIC CONSTRAINED PROGRAMMING

A. Prékopa

We consider the nonlinear programming problem

$$(1) \quad \text{minimize} \quad f(x)$$

subject to the conditions

$$(2) \quad P(Ax \geqq \beta) \geqq \alpha, \quad x \geqq 0.$$

This problem, called also chance constrained programming problem in the literature, is usually solved under the weaker conditions requiring

$$(3) \quad P\left(\sum_{k=1}^{n} a_{ik}x_k \geqq \beta_i\right) \geqq \alpha_i, \quad i = 1,\dots,m,$$

$$x_k \geqq 0, \quad k = 1,2,\dots,n,$$

and the reason for that is the possibility to prove in a relatively simple way the convexity of the set of feasible vectors. The use of (3) instead of (2) is not justified however, from the probabilistic point of view. If the rows in the system of inequalities $Ax \geqq \beta$ are independent, then the convexity of the set of vectors satisfying (2) can be lead back to the convexity problem of feasible vectors satisfying to the separate inequalities in (3). In many practical problems independence between rows cannot be supposed, however. First the case of a random β is considered (A is constant) and it is proved that if the components have multivariate normal distribution, then the set of the feasible vectors is convex for any correlation matrix, dimensions m,n if α is large enough. An example for a practical problem where such a situation occurs, is the nutrition problem where the nutrient requirements are correlated multivariate normal for a randomly chosen unit. Conditions are given for the convexity of the set of feasible vectors under random A and β too.

The solution procedure is outlined for the case of a convex objective function.

TUESDAY AFTERNOON

ITERATIVE SOLUTION OF LARGE, SPARSE LINEAR SYSTEMS AND RELATED TOPICS

L. D. Pyle and D. K. Smith

Iterative methods have been applied with considerable success in solving the large, linear systems $Ax = b$ which arise from discrete approximations to elliptic partial differential equations. Characteristically, such systems have coefficient matrices A which possess properties such as sparsity, band structure, diagonal dominance and non-negativity. Convergence proofs for the iterative methods used depend, in an essential way, on such properties as diagonal dominance or non-negativity. The practical utility of such methods depends both on rate of convergence and sparsity, the latter since the methods make implicit use of zero elements and thus only the relatively few non-zero elements require arithmetic modification during each iteration. For general sparse matrices which do not possess properties insuring convergence of standard iterative methods (such as Gauss-Seidel or Successive Overrelaxation), practical iterative methods are lacking.

There is a well-known method, due to Kaczmarz, which makes implicit use of sparsity in the iterative solution of arbitrary linear systems. Unfortunately, this method usually converges rather slowly. Altman and Pyle have each devised variations of the Kaczmarz method. Altman's variation is equivalent to solving $A^TAx = A^Tb$ by the Gauss-Seidel method, although A^TA and A^Tb need not be formed explicitly. Pyle's variation possesses a property which appears to be useful in connection with acceleration of convergence.

Algorithms for solving large, structured linear programming problems are also in short supply. The decomposition algorithm, due to Dantzig and Wolfe, appears to be the only algorithm available and it assumes a certain type of structure (block angularity) in the coefficient matrix. Little study has apparently been devoted to the iterative solution of the succession of linear systems which result when the simplex method is applied directly.

Gradient projection algorithms, and other algorithms which depend upon generalized inverses, usually require the determination of vectors which lie in the intersection of certain linear manifolds. Although direct methods have traditionally been used to calculate such projections, it is possible to obtain them as limits of certain Kaczmarz-like iterative processes.

This paper is a progress report dealing with research in the areas described above, with particular attention being given to the acceleration of the underlying iterations. Explorations involving semi-iterative methods and the Wynn ε_n-Algorithm will be discussed.

MONDAY AFTERNOON

INTERIOR POINT METHODS FOR THE SOLUTION OF MATHEMATICAL PROGRAMMING PROBLEMS

J. D. Roode

For the mathematical programming problem $\max[f(x) | x \varepsilon R]$ where $f(x)$ is a function of the vector $x \varepsilon E_n$ and R is specified by a set of inequalities, $R = [x| f_i(x) \leqq 0, i \varepsilon I]$, I being some finite index set, a class of methods, to be called interior point methods, will be discussed. In an interior point method a sequence $\{x^k\}$ is constructed such that for all k, $x^k \varepsilon R^o = [x| f_i(x) < 0, i \varepsilon I]$, $f(x^k) < f(x^{k+1})$ and any point of accumulation of the sequence is a constrained stationary point of $f(x)$. Depending on the way in which the sequence of interior points is constructed, different classes of interior point methods can be distinguished. It will be shown that a number of known methods belong to a certain class of interior point methods--permitting a unified treatment of these methods and indicating along which lines new methods may be developed. In addition, other classes of interior point methods will be treated, presenting new methods in the field of nonlinear programming. Finally, the advantages of using an interior point method will be discussed.

MONDAY AFTERNOON

APPROXIMATE COMPUTATIONAL SOLUTION OF NONLINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS BY LINEAR PROGRAMMING

J. B. Rosen

This work is based on a well-known technique of applied mathematics. The desired solution is approximated by a linear combination of a finite set of chosen functions. The approximation is obtained by determining the coefficients of the functions so as to minimize the error in an appropriate sense. The accuracy of the approximation for a particular problem will depend on the choice of approximating functions, the number of such functions used, and the error norm chosen. The uniform norm is chosen here, and the coefficients are determined by an appropriate use of linear programming. The nonlinear parabolic partial differential equation leads to an LP problem which may be considered as having its cost row given parametrically as a function of time.

TUESDAY AFTERNOON

ON NONLINEAR OPTIMIZATION IN INTEGERS

T. L. Saaty

The purpose of the paper is to investigate optimization in integers of some elementary nonlinear expressions subject to equality constraints. The hope is also to include some known facts from number theory which may be useful in this pursuit. Two purposes are sought in this approach: 1) To generate some possible methods for tackling such simple problems and 2) to show the infeasibility of continuous approximations to the desired answers. No substantial generalizations of this approach are likely. However, they serve as useful tools for improving insight and understanding into this vast area.

TUESDAY EVENING

THE SHRINKING BOUNDARY ALGORITHM FOR DIOPHANTINE PROGRAMMING

R. M. Saunders and R. Schinzinger

The algorithm described in this paper was constructed to solve pure integer linear programming problems by an enumerative search technique which is patterned on discrete parallel translations of selected boundary planes.

Gomory's cutting plane methods of integer programming rely on generating new sets of constraints which trim the feasible region until a convex hull is produced which has an integral-feasible extreme vertex.[1] Land and Doig translate the objective function until a feasible point is found in the plane of the objective function.[2] The shrinking boundary method presents a new approach in that some of the boundary planes on the convex hull of the feasible region are subjected to stepwise shifts.

The slack variables which transform a set of inequality constraints in integer form into a set of equality constraints must be integral valued. A parallel shift of a boundary plane, which is equivalent to a change in the corresponding slack variable, will therefore proceed in discrete increments without thereby overlooking any integral-feasible points. While a search based on a scan of the feasible region by the shifting of boundaries constitutes a finite process it is not necessarily efficient until criteria are established as to which boundaries should be moved and when the search should be terminated.

The algorithm incorporates tests which reduce search space and effort. A hierarchy of variables based on the angular position of the boundary planes with respect to the objective function determines which planes are to be moved and in what order. Feasibility tests based on diophantine analysis rule out infeasible boundary positions. Sensitivity tests determine when no further improvement of the optimand is possible without introducing an additional constraint on the value of the optimand.

The shrinking boundary algorithm appears to be at ats best when the angle inscribed at the extreme vertex (in the continuous variable sense) is obtuse. It is exactly problems of this type which can so easily lead to false results with the search trapped at a local optimum.

Numerical examples illustrate the procedures described by the algorithm.

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1) See Ch. 8 in G. Hadley's "Nonlinear and Dynamic Programming," Addison-Wesley, 1964.
2) A. H. Land and A. G. Doig: "An Automatic Method of Solving Discrete Programming Problems," Econometrica, Vol. 28 (1960), pp. 497-520.

TUESDAY EVENING

SOME DUALITY THEOREMS FOR THE NON-LINEAR VECTOR MAXIMUM PROBLEM

P. Schönfeld

In this paper duality for the non-linear vector maximum problem (as introduced by Kuhn-Tucker) is considered. Let $f(x)$ and $g(x)$ be vector-valued concave differentiable functions. Denote by f_x, g_x their Jacobians and put $F(x) = f(x)-f_x \cdot x$, $G(x) = g(x)-g_x \cdot x$. Then the primal problem is to find the vector maxima of the set

$$D = \{d : f(x) \geqq d, \quad \text{for some} \quad x \in X\}$$

where

$$X = \{x : g(x) \geqq 0, \quad x \geqq 0\} .$$

The dual problem is to determine the vector minima of the set

$$H = \{h : vF(y) + uG(y) \leqq vh, \quad \text{for some } (u,v,y) \in Z\}$$

where

$$Z = \{z = (u,v,y) : vf_y + ug_y \leqq 0, \quad u \geqq 0, \quad v >> 0\}.$$

The main result is that under Kuhn-Tucker regularity of the primal constraints complement D = interior H.

With this result it is easy to prove that under Kuhn-Tucker regularity any proper vector maximum of D is a vector minimum of H, and any vector minimum of H is a proper vector maximum of D.

The argument is based on a lemma which gives a sufficient condition that

$$\phi(v) = \sup_{x \in X} vx$$

where X is a closed convex set is attained for some $x^\circ \in X$, i.e. $\phi(v) = vx^\circ$. This lemma is a special result emerging from the Fenchel theory of conjugate convex functions. An independent proof is offered here.

FRIDAY AFTERNOON

APPLICATION OF NONLINEAR PROGRAMMING AND BAYESIAN STATISTICS TO THE THEORY OF THE FIRM

L. E. Schwartz

This paper uses nonlinear mixed integer programming to determine the optimal production levels of a monopolistic multi-product firm. Production takes place under conditions of increasing returns to scale. The firm faces nonlinear Bayesian demand functions in its product markets, but costs are assumed determinate. The cost functions consist of fixed and variable components. In addition the decision-maker possesses a non-linear utility function which is dependent on profits. This formulation of the firm's objective function avoids many of the problems inherent in trying to specify what the firm tries to maximize. The analysis is static, but can be generalized easily to comparative statics, assuming independence in utility from period to period.

TUESDAY EVENING

AN ALGORITHM FOR LINEARLY CONSTRAINED NONLINEAR ESTIMATION WITH EXTENSIONS TO GENERAL NONLINEAR PROGRAMMING

D. F. Shanno

The general nonlinear estimation problem is to minimize a functional

$$(1.) \quad F(\theta_1,\dots,\theta_n) = \sum_{i=1}^{m} f_i^{\,2}(\theta_1,\dots,\theta_n),$$

when the f_i's are arbitrary nonlinear functions of $\theta_1,\ldots,\theta_n$ and $m \geq n$. Physical considerations often restrict the θ_j's to a region R determined by

$$(2.) \quad A\theta \geq b \quad ,$$

where A is a $p \times n$ matrix, θ is the vector $(\theta_1,\ldots,\theta_n)$ and b is an arbitrary p - vector.

If we let $g^{(i)}$ denote $\nabla F^{(i)}$, the gradient of F evaluated at $\theta = \theta^{(i)}$, and denote by J_i the Jacobian matrix

$$J_i = \left[\frac{\partial f_k}{\partial \theta_j}\right]_{\theta = \theta^{(i)}} \quad ,$$

an iterative algorithm proposed by Marquardt to minimize (1) is

$$(3.) \quad \theta^{(i+1)} = \left[J_i' J_i + \lambda I\right]^{-1} g^{(i)} \quad ,$$

where λ is an arbitrary scalar with $\lambda > o$.

This paper extends the range of λ to $\lambda > -\xi_1$, where ξ_1 is the smallest eigenvalue of the non-negative definite matrix $J_i' J_i$. A method for calculating an optimal λ at each step is then determined. It is then shown how the method of projection matrices developed by Rosen may be coupled with the vector (3) with optimal λ selection to produce a second order method for minimizing (1) subject to (2).

Finally, it is shown how the method may be applied to the problem of minimizing an arbitrary convex nonlinear object function subject to linear constraints.

TUESDAY AFTERNOON

ALGORITHMS FOR MINIMAL TREES AND SEMI-STEINER TREES BASED ON THE SIMPLEX METHOD

J. W. Suurballe

A 1-1 relationship between steps of the dual simplex method and a well-known combinatorial algorithm for minimal spanning trees can be established when a certain linear program and programming strategy are used. (By a programming strategy we mean a well-defined scheme for using constraints and changing bases, which is consistent with but added to the usual simplex method). The linear program and strategy used for the minimal spanning tree can be extended to include more general tree problems, in particular the so-called Semi-Steiner tree problem: Given the primary points P_i, $i = 1, \ldots, n<\infty$, and the secondary points S_i, $i = 1, \ldots, r<\infty$,[1] find a minimal cost tree spanning the primary points and an arbitrary number of secondary points, when tree cost is the sum of all arc lengths in the tree, plus a fixed cost c_i for each secondary point S_i used. (This is the usual minimal spanning tree when no secondary points are included.)

The dual simplex method and a certain programming strategy applied to the Semi-Steiner problem provides the insight for generating a combinatorial procedure, which corresponds 1-1 with steps of the simplex method, and contains the previously known minimal spanning tree algorithm as a special case.

WEDNESDAY AFTERNOON

APPLICATION OF LINEAR AND NONLINEAR PROGRAMMING IN OPTIMAL CONTROL OF NUCLEAR REACTORS

D. Tabak

Two basic problems connected with optimal control and management of Nuclear Reactors are considered:

(1) The optimal management of Nuclear Reactor fuels in a long range operation.

(2) Optimal shutdown control of Nuclear Reactors, for Xenon poisoning.

In problem (1), the reactor is symbolically represented as a feedback control system, where the state variables are the nuclide concentrations of materials considered. The control variables symbolize the discrete changes in the nuclide concentrations at refueling times. The problem is to find an optimal refueling policy over a period containing a finite number of refuelings. As an objective function one may pose the goal of minimizing the quantity of U-235 used, or maximizing the Pu output. The objective function was expressed both in linear and quadratic forms. The problem variables were the values of the control variables at refueling times. A set of linearized constraints, assuring proper operation of the reactor have been posed. One of the difficulties was the fact that the refueling times were unknown a priori. An iterational algorithm of sequential solutions of Linear and Quadratic Programming problems, is proposed and implemented in this paper. Convergence of the algorithm has been established computationally. Sensitivity analysis around the optimal values obtained, has also been performed.

In problem (2), a Nonlinear Programming (NLP) solution to find the optimal control law for the shutdown of a Nuclear Reactor, under Xenon poisoning, is attempted. The concentrations of Xenon and Iodine serve as state variables, and the neutron flux is the control variable. The goal is to minimize the Xenon peak after shutdown, so that one could be able to restart the reactor after a shorter time lapse. At first, an optimal terminal condition for the state variables, to minimize the Xenon peak is obtained by direct differentiation. Then, the state equations are written in discrete form and a minimal time problem is solved using the SUMT NLP program, originated by Fiacco and McCormick. The minimal time obtained using this method, turns out to be much shorter than the time obtained using the maximum principle, for the same data.

WEDNESDAY AFTERNOON

A NEW ALGORITHM FOR TRANSPORTATION AND ASSIGNMENT PROBLEMS

G. L. Thompson

The first step in the algorithm is to employ a regret heuristic to construct an initial solution which, empirically, is better than the VAM start. Then the algorithm proceeds in the reverse direction gradually constructing the dual solution. From time to time it is necessary to use the regret heuristic to resolve subproblems. Advantages over the MODI method are: (a) the regret heuristic is fast and very good, (b) the solution of sub-problems gives information on the overall solution at a decrease in overall effort. Computational experience will be discussed.

MONDAY MORNING

THE SOLUTION OF PROGRAMMING PROBLEMS BY GENERATING FUNCTIONS

J. K. Thurber

A method has been developed for solving integer programming problems by using generating functions. By means of the generating functions the feasible solutions for a system with linear constraints can be represented in terms of new sets of integer variables which satisfy only the trivial constraints that they be non-negative. The minimization problem then becomes a triviality in principle although in practice it may lead to a tedious computation. Examples are included where both situations occur (i.e. where the method gives quick solutions and where it is impractical).

The procedure is based on an extension and simplification of some ideas of P. A. MacMahon on the theory of partitions. In addition the method leads to some results in multiplicative number theory, greatly generalizing some well known identities involving the Riemann zeta function.

The algorithm also has a continuous analog which amounts to solving simultaneous linear inequalities by means of Laplace transforms. The representation of the feasible solutions by means of Laplace transforms also eliminates all the non-trivial constraints and permits the minimization problem to be dealt with via classical methods of the calculus for non-linear objective functions and reduces the case of linear objective functions to a trivial although possibly tedious computation.

TUESDAY EVENING

ON SOLVING SOME CLASSES OF INTEGER LINEAR PROGRAMMING PROBLEMS

C. A. Trauth, Jr. and R. E. D. Woolsey

Some classes of integer linear programming problems exhibit a certain inefficient solution pattern when an attempt is made to solve them using the Gomory cutting-plane methods.

A technique has been developed which alleviates the inefficiency in calculations. It has, so far, led to more rapid solutions of problems possessing this inefficiency characteristic.

Although some mathematical insight into the nature of this technique has been gained, the study to date has been primarily empirical, and the results of using this technique will be discussed.

FRIDAY AFTERNOON

AN EXTENDED DUALITY THEOREM FOR CONTINUOUS LINEAR PROGRAMMING PROBLEMS

W. F. Tyndall

In A Duality Theorem for a Class of Continuous Linear Programming Problems (J. Soc. Indust. Appl. Math. 13 (1965) 644-666) we proved the THEOREM: Hypotheses: (I) $\{x \varepsilon E^N: \ Bx \leqq 0 \text{ and } x \geqq 0\} = \{0\}$, (II) B, C, and $c(t)$ have nonnegative components for $t \varepsilon [0,T]$. Conclusion: There exist optimal solutions $\overline{z}$ and $\overline{w}$ to the programs: maximize $\int_0^T z(t)\cdot a(t)\,dt$ subject to $z(t) \geqq 0$ and $Bz(t) \leqq c(t) + \int_0^t Cz(s)\,ds$, and minimize $\int_0^T w(t)\cdot c(t)\,dt$ subject to $w(t) \geqq 0$ and $w(t)B \geqq a(t) + \int_t^T w(s)C\,ds$. Furthermore, two feasible functions z and w are optimal if and only if $\int_0^T z(t)\cdot a(t)\,dt = \int_0^T w(t)\cdot c(t)\,dt$. In that theorem the vector-valued functions a and c were assumed to be continuous.

EXTENDED THEOREM: The previous theorem is valid even if the functions a and c are assumed to be only bounded and measureable. The key result needed for this extension is the LEMMA: Under Hypotheses (I) and (II) given a bounded, measureable function w satisfying the constraints for the minimum program only almost everywhere, there exists a bounded, measureable function $\hat{w}$ which satisfies the constraints for all $t \varepsilon [0,T]$ with $\int_0^T \hat{w}(t)\cdot c(t)\,dt \leqq \int_0^T w(t)\cdot c(t)\,dt$. (The corresponding result for the maximum program is trivial.)

WEDNESDAY AFTERNOON

THE GUIDING PERMUTATION METHOD FOR COMBINATORIAL PROBLEMS

E. Valensi

This paper deals with combinatorial problems in which a set of choices have to be made, subject to a given set of constraints. For instance, in the travelling salesman problem with n cities, the choices consist in retaining or rejecting each of the $n(n-1)/2$ edges of the graph so as to construct a hamiltonian cycle.

For real problems, the number of feasible solutions may be of the order of $200\,!$ or 2^{1000}. A standard branch and bound method is then, not feasible, since, whatever the rules of branching and the bounding function are, the tree to be explored is far too large. To reduce the size of the tree, it is possible either to truncate it, which leads to a heuristic procedure (conditions will be given which allow iteration of truncated branch and bound methods), or to cut down the number of solutions by introducing necessary optimality conditions.

It is convenient, for the study of truncation to introduce a "guiding permutation" G. G is the fixed sequence, identical for each branch of the tree, in which choices are made. We call "main heuristic solutions" corresponding to a guiding permutation G, the solution $\mathcal{H}(G)$ obtained by locally optimizing the economic function f(S) in the order imposed by G.

For a maximization problem, under quite general conditions, and suitably chosen truncation rules, we show that:

a) The solution $\mathcal{S}(G)$ obtained by a truncated branch and bound method corresponding to a guiding permutation G **verifies:**

$$f(\mathcal{S}(G)) \geq f(\mathcal{H}(G))$$

b) To each solution $S = \mathcal{S}(G)$ can be associated a guiding permutation (S) such that:

$$\mathcal{H}(\mathcal{G}(S)) = S$$

Then

$$f(\mathcal{S}(\mathcal{G}(S))) \geq f(\mathcal{H}(\mathcal{G}(S))) = f(S)$$

This inequality shows that interating the composed operation leads to a sequence of non-decreasing solutions.

To define necessary optimality conditions is equivalent to define a subset T of the set of solutions Σ in which the optimum is sure to be.

When searching for a permutation, i.e. an ordering, a partial ordering with which the optimum order has to be consistent gives such a necessary condition.

Let us suppose that a distance $d(S_1, S_2)$ between solutions has been defined, that a heuristic solution S' has been found and that we are able to find an upper-bound E(S') of its distance to the optimum S*, that is:

$$d(S', S^*) \leq E(S') ;$$

X being a set of solutions, we suppose also we can find a lower bound e(S', X) of the distance d(S', S) for $S \in X$:

$$\forall \quad S \in X , \quad e(X', X) \leq d(S', S)$$

Then the relation

$$e(S', X) \leq E(S')$$

definies an optimality condition.

Two examples will be given.

TUESDAY EVENING

STOCHASTIC PROGRAMS WITH RECOURSE: SPECIAL FORMS

D. W. Walkup and R. J.-B. Wets

In a previous paper the authors have defined stochastic programs with recourse and developed some of their theoretical properties. (Briefly a stochastic program with recourse arises when essentially all parameters are allowed to be random in the two-stage programming under uncertainty model introduced by G. Dantzig.) The objective of the present paper is twofold. First, to identify, and set out some of the properties of, certain special cases of stochastic programs with recourse such as: (i) fixed recourse, the matrix W of second-stage activity coefficients is nonrandom; (ii) simple recourse, the matrix W is [I,-I], a generalization of a model studied by Williams and Wets; and (iii) stable recourse, W is square, a special case related to the so-called distribution problem. Second, to show how some stochastic programs, such as the active approach due to G. Tintner, which at first glance do not seem to fall into the framework of stochastic programs with recourse, can be put in that form.

TUESDAY AFTERNOON

MINIMIZING A CONVEX FUNCTION OVER A CONVEX HULL

R. J.-B. Wets and C. Witzgall

Let $S = \{A_1, \ldots, A_n\}$ be a finite collection of points in R^m, and let $f(x)$ be a real valued differentiable convex function defined on R^m. We consider the two problems:

Problem 1: Minimize $f(x)$ over the convex hull con(S) of S.

Problem 2: Minimize $f(x)$ over the positive hull pos(S) of S.

There are two obvious routes of attack: cutting plane techniques or feasible directions. We choose the latter, making use of the special structure of the problem. The resulting algorithm will combine pivoting in a matrix, whose columns correspond to the points in S, with a method of unconstrained minimization. The latter can be selected arbitrarily from a class, which we call "gradient restriction methods", and which comprises Steepest Descent, Fletcher-Powell, Newton-Raphson, Univariant Minimization, and related methods.

At each stage of an algorithm, the current location x will be represented as a combination of generators A_i, spanning a simplex (resp. simplicial cone) Δ. Before leaving Δ the representation is changed, which corresponds to the pivot steps mentioned above. The pivoting rules follow from the authors' results on the algebraic characterization of the face structure of convex polyhedral cones.

The optimality criterion is as follows: point x is optimal if the contour tangent at x supports con(S) (resp. pos(S)),--a fact which can be verified simply by examining the finitely many points in S. The gradient is then used in a double role: for describing the contour tangent needed for the optimality criterion, and for obtaining a new direction, both feasible and profitable, along which to proceed.

Problem 2 arises in Stochastic Programming, and the algorithm seems to be particularly well adapted to solve Stochastic Programs with Simple Recourse, and their generalization by Williams to the nonlinear case. In particular, it can be seen how the algorithm takes advantage of the approximation formulas developed by Williams for Stochastic Programs with Simple Recourse.

THURSDAY AFTERNOON

PROGRAMMING UNDER UNCERTAINTY: THE LINEAR RECOURSE PROBLEM

A. C. Williams

We consider a linear program in which the right-hand side (the supply/demand vector in an activity analysis model) is a random variable, and in which linear penalties or salvage values may be applied in case of under or over production. It is assumed that we wish to choose an x so as to maximize the expected profit function. This problem has been called the complete problem by R. Wets; we call it the linear recourse problem, the terminology suggested elsewhere by Wets and D. Walkup. It is perhaps the simplest of the stochastic models that have been considered, but it is of interest because (1) it is a genuine generalization of ordinary linear programming; (2) manageable existence and characterization theorems can be given, (3) given a solution of an ordinary linear program, we may estimate how good it is as an approximate solution to the linear recourse problem; and (4) the case of convex penalty costs can be approximated as closely as we please by linear recourse problems.

Certain results on these problems have recently been obtained by us [1], [2], by A. Charnes, M. Kirby, and W. Raike [3], by Wets [4], and by R. Wilson [5]. In this paper we show that the recent work of R. T. Rockafellar [6] on conjugate function theory can be applied to obtain essentially all of these results (with the exception of the algorithmic development of Wets). Moreover, we extend significantly the duality results of [5], and strengthen the existence, characterization and approximation results of [1] and [2].

REFERENCES:

[1] Williams, A. C., "On Stochastic Linear Programming, SIAM Journal 13 (1965), pp. 927-940.

[2] _______________ , "Approximation Formulas for Stochastic Linear Programming," SIAM Journal 14 (1966), pp. 668-677.

[3] Charnes, A., Kirby, M. J. L., and Raike, W. M., "On a Special Class of Constrained Generalized Median Problems," Systems Research Memo #155, Northwestern University (1966).

[4] Wets, R. J. B., "Programming Under Uncertainty: The Complete Problem," Z. Wahrscheinlichkeitstheorie Verw. Gebiete 4 (1966), pp. 316-339.

[5] Wilson, R., "On Linear Programming Under Uncertainty," Operations Research 14 (1966), pp. 652-657.

[6] Rockafellar, R. T., "Convex Programming and Systems of Elementary Monotonic Relations," to appear in J. Math. Anal. Appl.

TUESDAY AFTERNOON

GAMES, LINEAR PROGRAMMING AND THE TCHEBYCHEFF APPROXIMATION

L. B. Willner

The relations between these three topics are developed using new formulations of the equivalence between matrix games and the dual linear programs which correspond to discrete Tchebycheff approximation problems. This approach leads to a very simple correspondence between matrix games and discrete Tchebycheff approximation problems. It is shown that while the dual linear programs corresponding to a matrix game need not represent Tchebycheff approximation problems, the game is nevertheless equivalent to a dual pair of Tchebycheff problems. Further it is shown that every Tchebycheff problem has a corresponding game.

TUESDAY EVENING

INTERCEPTION IN A NETWORK

R. D. Wollmer

This problem involves a game between two players, an evader and a pursuer, who are constrained to operate on a network of nodes and arcs. The evader wishes to choose a path from a source node in the network to a sink node in the network in such a way that his probability of successful traverse is maximized. The pursuer has N forces which he can place on nodes and arcs with the intention of minimizing the maximum evader escape probability. Only the pursuer's strategy is developed in this paper. Each node i has a number $p(i)_k$ representing the probability the evader will be stopped at node i if it is on his chosen path and the pursuer has k of his forces placed there. Each arc (i,j) has similarly defined probabilities $p(i,j)_k$ associated with it.

The $p(i)_k$ and $p(i,j)_k$ are assumed to satisfy the law of diminishing returns. The pursuer's problem is thus one of choosing numbers $\pi(i)$ and $\pi(i,j)$, representing the expected number of forces he will place at the nodes and arcs, such that their sum is at most N, and such that the evader's maximum probability source sink path is minimized. While this problem can theoretically be solved by constructing the game matrix, the number of pure strategies for moderate sized networks can be extremely large.

This paper presents incremental approach which is much shorter and reduces the computation to solving a sequence of maximum flow problems. The algorithm found yields an optimal solution for $N=1$, and yields a good approximation for $N>1$.

WEDNESDAY AFTERNOON

ON A PRIMAL INTEGER PROGRAMMING ALGORITHM

R. D. Young

This paper describes a primal, all-integer algorithm for solving a bounded and solvable pure integer programming problem. The method is a primal analogue to the Gomory All-Integer Algorithm, and is a variant of the simplex method in the sense that the Gomory algorithm is a variant of the dual method. The simplified primal algorithm includes these major amendments to the simplex method: (i) a special row, indexed by L, is adjoined to the tableau and is periodically revised by a well-defined procedure; (ii) in most cycles of the algorithm the pivot column, A_J, is selected so that $a_{LJ} > 0$ and $(1/a_{LJ})A_J$ is lexicographically smaller than $(1/a_{Lj})A_j$ for all other non-basic columns A_j that have a $a_{Lj} > 0$; (iii) in all cycles of the algorithm a Gomory cut is adjoined after selection of the pivot column, and the cut is selected so that it will have a unit coefficient in the pivot column and it will qualify (in order to be used) as the pivot row. With comparatively weak restriction on the selection of the row used to generate the Gomory cut the simplified primal algorithm is shown to be finite.

FRIDAY AFTERNOON

APPLICATIONS OF THE CONVERGENCE CONDITIONS

W. I. Zangwill

In a previous paper this author posed necessary and sufficient conditions for the convergence of algorithms. In this paper these conditions are specialized and the relation of the upper semi-continuous mapping developed. Exploitation of these tools provides straightforward convergence proofs for many algorithms. As examples, algorithms, by Cauchy, Newton, Huard, and Frank and Wolfe are established.

TUESDAY EVENING

ENUMERATION ALGORITHMS FOR PURE AND MIXED INTERGER PROGRAMMING

G. Zoutendijk

The main subject of this paper is the description of an enumeration algorithm for the 0-1 pure integer programming problem. This algorithm is partially based on exclusion rules and selection criteria developed by Benders, Balas and Glover but a number of new rules have been added. Initial computational experience has been encouraging. Next it is shown how this algorithm, in a slightly modified form, can efficiently be used to solve the integer sub-problems which arise when dual decomposition is applied to the solution of the mixed integer linear programming problem of the 0-1 type.

As an alternative to decomposition direct tree-searching can be applied to solve the mixed problem. The efficiency of the search is heavily dependent upon the structure of the tree and the way it is being built. A particular way will be demonstrated which is supposed to be more effective than previous proposals.

FRIDAY AFTERNOON

MONDAY, AUGUST 14

REGISTRATION: Princeton Playhouse,
9 a.m. to 10 a.m.

OPENING SESSION: 10 a.m. to 12 noon

DEAN L. SPITZER, JR., Princeton University
Welcoming Remarks

G. B. DANTZIG
Invited Survey: Large Scale System

E. M. L. BEALE
Matrix Generators and Output Analyzers

PARALLEL SESSIONS: 1:45 p.m. to 4:30 p.m.

SESSION A: LARGE SCALE SYSTEMS: McCosh 10
Chairman: A. C. WILLIAMS

J. ABADIE and M. SAKAROVITCH
A Method of Decomposition

L. D. PYLE and D. K. SMITH
Iterative Solution of Large, Sparse Linear Systems

W. ORCHARD-HAYS
The Block Product Decomposition Algorithm

R. H. COBB and J. CORD
A Decomposition Approach for Solving Linked Programs

T. O. M. KRONSJO
Centralization and Decentralization of Decision Making: The Double Decomposition Method — Generalization and Proof of Convergence

SESSION B: ALGORITHMS: Frick 138
Chairman: A. W. TUCKER

A. BEN-ISRAEL
On Newton's Method in Nonlinear Programming

H. D. MILLS
Extending Newton's Method to Systems of Inequalities

J. D. ROODE
Interior Point Methods for Mathematical Programming Problems

V. NALBANDIAN
The Shifting-Objective Algorithm

G. L. THOMPSON
A New Algorithm for Transportation and Assignment Problems

EVENING SESSION: McCosh 10, 8 p.m.

PANEL DISCUSSION OF THE DESIGN AND DEVELOPMENT OF MATHEMATICAL PROGRAMMING SYSTEMS
Chairman: E. M. L. BEALE
D. W. HALLENE, M. MINNS, C. A. HAVERLY

TUESDAY, AUGUST 15

GENERAL SESSION: Princeton Playhouse, 9 a.m. to 12 noon
Chairman: J. ABADIE

P. WOLFE (MAY)
Invited Survey: Nonlinear Programming

G. ZOUTENDIJK
On Continuous Finite Dimensional Constrained Optimization

R. T. ROCKAFELLAR (MAY)
Conjugate Convex Functions in Nonlinear Programming

A. R. COLVILLE
A Comparative Study of Nonlinear Programming Codes

PARALLEL SESSIONS: 1:45 p.m. to 5 p.m.

SESSION A: A PROGRAMMING UNDER UNCERTAINTY: McCosh 10
Chairman: R. M. THRALL

M. J. L. KIRBY
The Current State of Chance-Constrained Programming

D. W. WALKUP and R. J. B. WETS
Stochastic Programs with Recourse: Special Forms

A. C. WILLIAMS
Programming Under Uncertainty: The Linear Recourse Problem

A. PREKOPA
On Probabilistic Constrained Programming

B. BEREANU
Renewal Processes and Some Stochastic Programming Problems

SESSION B: ALGORITHMS: Frick 138
Chairman: P. WOLFE

LRB A. V. FIACCO and G. P. McCORMICK
Nonconvex and Convex Programming by Generalized Sequential Methods

M. COURTILLOT
A Method for Convex Programming

R. E. CLINE and L. D. PYLE
On the Solution of Structured LP Problems using Generalized Inverses

D. F. SHANNO
An Algorithm for Linearly Constrained Nonlinear Estimation

J. B. ROSEN
Approximate Computational Solution of Nonlinear Parabolic Partial Differential Equations by Linear Programming

EVENING PARALLEL SESSIONS: 8 p.m.

SESSION A: McCosh 10
Chairman: R. H. COBB

T. L. SAATY
On Nonlinear Optimization in Integers

J. MORAVEK
On the Complexity of Integer Programming Problems

R. M. SAUNDERS and R. SCHINZIGER
The Shrinking Boundary Algorithm for Diophantine Programming

A. M. GEOFFRION
Implicit Enumeration Using an Imbedded Linear Program

J. K. THURBER
The Solution of Programming Problems by Generating Functions

E. VALENSI
The Guiding Permutation Method for Combinatorial Problems

J. R. HEMSLEY
A Class of Finite-Time Markov-Renewal Programming Situations

L. E. SCHWARTZ
Application of Nonlinear Programming and Bayesian Statistics to the Theory of the Firm

SESSION B: Frick 138
Chairman: R. W. COTTLE

V. DeANGELIS
Minimization of a Separable Function Subject to Linear Constraints

P. BOD
The Solution of a Fixed Charge Linear Programming Problem

O. L. MANGASARIAN
Optimality and Duality in Nonlinear Programming

L. B. WILLNER
Games, Linear Programming and the Tchebycheff Approximation

LRB W. I. ZANGWILL (MAY)
Applications of the Algorithmic Convergence Conditions

L. HALLER
LRB
A Hypercone Search Optimization Method

M. AVRIEL and D. J. WILDE
Stochastic Geometric Programming

V. J. LAW and R. H. FARISS
Nonlinear Programming via Rotational Discrimination

WEDNESDAY, AUGUST 16

GENERAL SESSION: Princeton Playhouse, 9 a.m. to 12 noon
Chairman: J. B. ROSEN

M. CANON
Invited Survey: Control Theory and Mathematical Programming

H. HALKIN
Programming in Infinite Dimensional Space and Control Theory

F. SUPNICK
The Traveling Salesman Problem

PARALLEL SESSIONS: 1:45 p.m. to 6 p.m.

SESSION A1: CONTROL THEORY: McCosh 10, 1:45 p.m. to 3:45 p.m.
Chairman: H. HALKIN

W. P. DREWS and R. G. SEGERS
Continuous Mathematical Programming under Linear Integral Constraints

D. TABAK
Application of Linear and Nonlinear Programming in Optimal Control of Nuclear Reactors

G. HORNE and G. S. TRACZ
Nonlinear Programming and Second-Variation Schemes

SESSION A2: THEORY AND DUALITY: McCosh 10, 4:15 p.m. to 6 p.m.
Chairman: C. E. LEMKE

A. CHARNES and K. KORTANEK
On Classes of Convex and Preemptive Nuclei for n-Person Games

R. J. DUFFIN and E. L. PETERSON
Recent Developments in Geometric Programming

G. E. BLAU and D. J. WILDE
Second Order Characterization of Generalized Polynomial Programs

W. F. TYNDALL
An Extended Duality Theorem for Continuous Linear Programming Problems

SESSION B: NETWORKS: Frick 138,
Chairman: G. L. THOMPSON

T. C. HU
A Decomposition Algorithm for Shortest Paths in a Network

D. R. FULKERSON
The Max-Flow Min-Cut Equality and the Length-Width Inequality for Real Matrices

M. L. BALINSKI
On Maximum Matching, Minimum Covering, and Duality

E. L. JOHNSON, J. EDMONDS and S. LOCKHART
The Degree-Constrained Subgraph Problem

V. E. BENES
Optimal Routing in Connecting Networks over Finite Time Intervals

J. W. SUURBALLE
Algorithms for Minimal Trees and Semi-Steiner Trees

O. I. FRANKSEN
Mathematical Programming by Physical Analogies

R. D. WOLLMER
Interception in a Network

EVENING SESSION: McCosh 10, 8 p.m.

SPECIAL INTEREST GROUP IN MATHEMATICAL PROGRAMMING (SIGMAP): Report on activities, including three recent Workshops on Branch and Bound, Unconstrained Optimization, and Programming System Development

THURSDAY, AUGUST 17

GENERAL SESSION: Princeton Playhouse, 9 a.m. to 12 noon
Chairman: R. R. SINGLETON

H. W. KUHN
Invited Survey: Mathematical Programming and Economic Theory

H. SCARF
On the Computation of Equilibrium Prices

D. GALE
Optimal Economic Development: A Concave Programming Problem in Denumerably Many Variables

PARALLEL SESSIONS: 1:45 p.m. to 6 p.m.

SESSION A1: THEORY AND DUALITY: McCosh 10,
1:45 p.m. to 3:45 p.m.
Chairman: D. R. FULKERSON

E. G. GOLSHTEIN
Dual Convex and Fractional-Convex Programming Problems

J. EDMONDS
Matroids and Extremal Combinatorics

C. P. BRUTER
Orthogonality in Matroids and Mathematical Programming

V. KLEE
Recent Results on the Combinational Structure of Convex Polytopes

SESSION B1: NONLINEAR PROGRAMMING ALGORITHMS: Frick 138,
1:45 p.m. to 3:45 p.m.
Chairman: H. W. KUHN

R. J. B. WETS and C. WITZGALL
Minimizing a Convex Function over a Convex Hull

G. ZOUTENDIJK
Enumeration Algorithms for Pure and Mixed Integer Programming

W. MURRAY
LRB
Ill-conditioning Occurring in Penalty and Barrier Functions

SESSION A2: ECONOMIC APPLICATIONS:
McCosh 10,
4:15 p.m. to 6 p.m.
Chairman: H. SCARF

R. BEALS and T. C. KOOPMANS
Maximizing Stationary Utility in a Constant Technology

D. McFADDEN
On the Existence of Optimal Programs of Economic Growth

A. K. KLEVORICK
L.RB
Mathematical Programming and Project Interrelationships in Capital Budgeting

G. CASALE
A Theoretical Analysis of Inputs Taxation under Linear Programming Assumptions

SESSION B2: APPLICATIONS: Frick 138,
4:15 p.m. to 6 p.m.
Chairman: H. D. MILLS

B. MEISTER and W. OETTLI
On the Capacity of a Discrete, Constant Channel

A. J. HOFFMAN and T. J. RIVLIN
When is a Team "Mathematically" Eliminated?

L. S. LASDON and A. D. WAREN
LRB
Nonlinear Programming and Engineering Design

J. P. KOHLI and F. W. LEAVITT
Optimum Design of Long Pipe Line Networks

BANQUET: Princeton Inn, 7:30 p.m.
Speaker: F. J. WEYL,
National Academy of Sciences

FRIDAY, AUGUST 18

GENERAL SESSION: Princeton Playhouse, 9 a.m. to 12 noon
Chairman: R. L. GRAVES

M. L. BALINSKI
Invited Survey: Integer Programming

R. E. GOMORY
Faces of an Integer Polyhedron

E. BALAS
Duality and Pricing in Discrete Programming

PARALLEL SESSIONS: 1:45 p.m. to 3 p.m.

woodrow wilson
SESSION A: INTEGER PROGRAMMING: ~~McCosh~~ 10
Chairman: M. L. BALINSKI

R. E. D. WOOLSEY and C. A. TRAUTH, JR.
On Solving Some Classes of Integer Linear Programming Problems

P. HUARD
Programmes Mathematique Nonlineaires a Variables Bivalentes

R. D. YOUNG
On a Primal Integer Programming Algorithm

SESSION B: NONLINEAR PROGRAMMING THEORY: Frick 138
Chairman: G. ZOUTENDIJK

P. SCHONFELD no address available
Some Duality Theorems for the Non-Linear Vector Maximum Problem

M. GUIGNARD
On the Kuhn-Tucker Theory

K. FAN
Asymptotic Cones and Duality of Linear Relations

SPECIAL SESSION: McCosh 10, 3:30 p.m. to 5 p.m.
PIVOTAL METHODS AND COMPLEMENTARY SOLUTIONS
Chairman: G. B. DANTZIG

A. W. TUCKER, C. E. LEMKE, R. W. COTTLE, T. D. PARSONS, R. L. GRAVES